# SpecRidge: A spectral Ridge benchmark for pressure-velocity Stokes Solvers 

Marc Spiegelman

Lamont-Doherty Earth Obs. of Columbia University Dept. of Applied Physics and Applied Mathematics, Columbia Univ.

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## 1 Introduction

Magma Dynamics as described by McKenzie [1984] and others is most easily understood as a consistent coupling of Darcy's law for fluid flow to Stokes flow for the viscously deforming solid phase [see Spiegelman et al., 2006]. An important aspect of this coupling, however, is that viscous deformation of the solid can lead to dynamic pressure gradients that can affect the flow of melt [e.g. Phipps Morgan, 1987, Spiegelman and McKenzie, 1987]. Thus it is imperative that Stokes solvers for solid flow return accurate pressures (and more importantly pressure gradients).

Here we present a useful pressure and flow benchmark for Stokes solvers that is relevant to the problem of melt migration beneath a mid-ocean ridge where the divergent flow of solid near the ridge axis can produce large pressure gradients if mantle viscosities are greater than $\sim 10^{20}$ Pa s. While there is still considerable debate as to the appropriate rheology of the mantle in the near sub-ridge region, a useful place to begin is iso-viscous pressure and velocity solutions as these can be calculated rapidly in 2- and 3-D to spectral accuracy. This document lays out the basic solutions and presents several matlab codes for calculating benchmark solutions.

## 2 General Equations for Incompressible Iso-viscous Stokes

Here we just consider boundary driven flow of an iso-viscous, incompressible fluid with an arbitrary upper velocity boundary condition over an infinite half-space in cartesian coordinates (these problems are easily extended to layered systems etc.).

In the absence of body forces, incompressible iso-viscous Stokes flow can be written

$$
\begin{gather*}
\boldsymbol{\nabla} \cdot \mathbf{V}=0  \tag{1}\\
\boldsymbol{\nabla} P=\eta \nabla^{2} \mathbf{V} \tag{2}
\end{gather*}
$$

or taking the Curl of Eq. (2) to remove pressure

$$
\begin{equation*}
\boldsymbol{\nabla} \times \nabla^{2} \mathbf{V}=\mathbf{0} \tag{3}
\end{equation*}
$$

With fixed velocity boundary conditions at $z=0$ (the surface) and $z \rightarrow \infty$. Assuming cartesian geometry, we can take the Fourier transform of Eqs. (1) and (3) in the horizontal to yield a system of equations in $z$ and horizontal wavevectors $\mathbf{k}=k_{x} \mathbf{i}+k_{y} \mathbf{j}$ For example, Eq. (1) becomes

$$
\begin{equation*}
i k_{x} \tilde{U}+i k_{y} \tilde{V}+D \tilde{W}=0 \tag{4}
\end{equation*}
$$

where $\tilde{\mathbf{V}}(\mathbf{k}, z)=\tilde{U} \mathbf{i}+\tilde{V} \mathbf{j}+\tilde{W} \mathbf{k}$ is the horizontal Fourier transform of the velocity field and $D=\frac{d}{d z}$ is the derivative operator in the vertical. Likewise, Eq. (3) becomes

$$
\begin{align*}
\left(D^{2}-k^{2}\right)\left(i k_{y} \tilde{W}-D \tilde{V}\right) & =0  \tag{5}\\
\left(D^{2}-k^{2}\right)\left(i k_{x} \tilde{W}-D \tilde{U}\right) & =0  \tag{6}\\
\left(D^{2}-k^{2}\right)\left(i k_{x} \tilde{V}-i k_{y} \tilde{U}\right) & =0 \tag{7}
\end{align*}
$$

where $k^{2}=\mathbf{k}^{T} \mathbf{k}$ and $\left(D^{2}-k^{2}\right)$ is the Fourier transform of the Laplacian $\nabla^{2}$. Equations (4)-(7) can be rearranged to form a single 4th order constant coefficient ODE for $\tilde{W}$ which is solved subject to boundary conditions and then transformed back to the space domain. Phipps Morgan and Forsyth [1988] used this approach to produce some of the first 3-D solutions for the flow near a ridge-transform-ridge intersection, and a similar 2-D spectral solution was used in Spiegelman [1996] to calculate melt and solid flow beneath ridges. One of the main advantages of the spectral solution over the standard corner-flow solution [e.g. Batchelor, 1967], is that the corner-flow solution has a singularity in the pressure due to the discontinuity in velocity at the ridge axis which makes it difficult to accurately calculate melt flow. However, the spectral solution allows one to control the strength of the pressure singularity by smoothing out the velocity at the ridge axis and producing numerically resolvable solutions.

We will demonstrate the solution and approach first in 2-D then use the 2-D solution to consider the affect of 3-D ridge geometries as in Phipps Morgan and Forsyth [1988].

## 3 2-D pressure, velocity solutions

In 2-dimensions, we let $\mathbf{V}(x, z)=U \mathbf{i}+W \mathbf{k}$ and choose boundary conditions

$$
\begin{equation*}
\mathbf{V}(x, 0)=U_{0}(x) \mathbf{i} \quad \mathbf{V}(x, \infty)=\mathbf{0} \tag{8}
\end{equation*}
$$

such that the flow is driven entirely by the upper boundary. In 2-D Eqs. (4)-(7) reduce to

$$
\begin{gather*}
i k \tilde{U}+D \tilde{W}=0  \tag{9}\\
\left(D^{2}-k^{2}\right)(i k \tilde{W}-D \tilde{U})=0 \tag{10}
\end{gather*}
$$

where $k=k_{x}$. These can be reduced to a single ODE in $\tilde{W}$ by multiplying Eq. (9) by $i k$, Eq. 10) by $D$ and eliminating $\tilde{U}$ to yield

$$
\begin{equation*}
\left(D^{2}-k^{2}\right)^{2} \tilde{W}=0 \tag{11}
\end{equation*}
$$

which has general solution

$$
\begin{equation*}
\tilde{W}(k, z)=\left(c_{1}+c_{2} z\right) e^{-|k| z}+\left(c_{3}+c_{4} z\right) e^{|k| z} \tag{12}
\end{equation*}
$$

where $|k|=\sqrt{k_{x}^{2}}=| | \mathbf{k} \|$ is positive to guarantee consistent behavior of all wavenumbers with depth. Applying the boundary conditions that the vertical velocity vanishes at $z=0$ and $z \rightarrow \infty$ (i.e. $\tilde{W}(k, 0)=\tilde{W}(k, \infty)=0$ ). Then

$$
\begin{equation*}
\tilde{W}(k, z)=c_{2} z e^{-|k| z} \tag{13}
\end{equation*}
$$

Applying Eq. (9) gives

$$
\begin{equation*}
D \tilde{W}=-i k \tilde{U}=c_{2}(1-|k| z) e^{-|k| z} \tag{14}
\end{equation*}
$$

and the boundary condition $\tilde{U}(k, 0)=\tilde{U}_{0}$ gives $c_{2}=-i k \tilde{U}_{0}$. Finally, applying $\boldsymbol{\nabla} P=\nabla^{2} \mathbf{V}$ gives the full solution in the Fourier domain for velocity and pressure

$$
\begin{align*}
\tilde{U} & =\tilde{U}_{0}(1-|k| z) e^{-|k| z}  \tag{15}\\
\tilde{W} & =-i k \tilde{U}_{0} z e^{-|k| z}  \tag{16}\\
\tilde{P} & =-2 i k \tilde{U}_{0} e^{-|k| z} \tag{17}
\end{align*}
$$

with individual components of the pressure gradient given by

$$
\begin{align*}
& \tilde{P}_{x}=2 \tilde{U}_{0}|k|^{2} e^{-|k| z}  \tag{18}\\
& \tilde{P}_{z}=2 \tilde{U}_{0} k|k| e^{-|k| z} \tag{19}
\end{align*}
$$

Transforming back to the space domain gives the full solution for velocity, pressure and $\nabla P$ to spectral accuracy.

### 3.1 Implementation

The matlab function specRidge2D implements the above algorithm given a horizontal velocity for the upper boundary. Because the discrete Fourier transform assumes periodicity, this function allows for a very large periodic domain for the boundary condition, but returns velocities and pressures on a smaller domain. Some care needs to be taken with the size and frequency content of the boundary conditions to avoid standard spectral issues such as Gibbs effects. A companion function setTopVelocity2D provides a useful example boundary condition to approximate ridge-spreading with an erf smoothed spreading rate profile (see Figure 1).


Figure 1: Example output from setTopVelocity2D to set upper horizontal velocity boundary condition appropriate for mid-ocean ridge spreading centers. All of these problems are scaled to the Extraction Length $L=\sqrt{\eta U_{0} / \Delta \rho g}$ [Spiegelman and McKenzie, 1987], which for this and the next figure we have taken to be 10 km . The periodic domain extends from $-24 L \leq x \leq 24 L$ and is discretized into 1024 panels of size $d x$. The velocity assumes an erf smoothed step function at $x=0$ $(U(x, 0)=\operatorname{erf}(x / \lambda))$ and tanh tapers at either end to enforce periodicity and remove spurious pressure oscillations due to Gibbs effects. Here $\lambda=0.1$.

## 4 3-D Benchmark

Extending the solution to 3-D for a Ridge-Transform-Ridge benchmark (e.g. Phipps Morgan and Forsyth 1988]) is straightforward and follows directly from the 2-D solution. Starting with Eqs. (4)-(7) we similarly reduce the problem to a 4th-order constant coefficient ODE in the vertical for $\tilde{W}(\mathbf{k}, z)$ and then use the boundary conditions at the surface to set the coefficients. Combining Eqs. (4)-(6) to eliminate $\tilde{U}$ and $\tilde{W}$ yields

$$
\begin{equation*}
\left(D^{2}-k^{2}\right)^{2} \tilde{W}=0 \tag{20}
\end{equation*}
$$

which is identical to Eq .11 but now $k=\|\mathbf{k}\|=\sqrt{k_{x}^{2}+k_{y}^{2}}$ which is always positive ${ }^{1}$. Applying boundary conditions that $W$ vanishes at $z=0$ and $z \rightarrow \infty$ again yields Eq. 13 for the general solution for $\tilde{W}$. To fix the value of $c_{2}$ however, requires both Eqs. (4) and (7) and the Fourier transform of the surface horizontal velocity fields $\tilde{\mathbf{U}}_{0}=\tilde{U}_{0} \mathbf{i}+\tilde{V}_{0} \mathbf{j}$ where $\tilde{U}_{0}=\tilde{U}(\mathbf{k}, 0)$. Substituting Eq. (13) into (4) gives

$$
\begin{equation*}
i k_{x} \tilde{U}+i k_{y} \tilde{V}=-c_{2}(1-k z) e^{-k z} \tag{21}
\end{equation*}
$$

and integrating Eq. 7) to remove $\left(D^{2}-k^{2}\right)$ (and assuming all velocities vanish as $z \rightarrow \infty$ gives

$$
\begin{equation*}
-i k_{y} \tilde{U}+i k_{x} \tilde{V}=c_{3} e^{-k z} \tag{22}
\end{equation*}
$$

where $c_{3}$ is an arbitrary constant of integration. Equations (21) and (22) can be written as

$$
\begin{equation*}
k Q \tilde{\mathbf{U}}=E(z) \mathbf{c} \tag{23}
\end{equation*}
$$

where

$$
Q=\frac{i}{k}\left[\begin{array}{cc}
k_{x} & k_{y}  \tag{24}\\
-k_{y} & k_{x}
\end{array}\right]
$$

[^0]

Figure 2: Example output from specRidge2D(produced by the script seeRidge2D) for the boundary condition given in Figure 1. The solution is returned only over the smaller domain $x \in[-6 L, 6 L]$ $z \in[0,6 L]$. In the lower two figures, a pressure gradient $>1$ denotes dynamic pressures greater than gravity. Note the very large pressure gradients on small scales generated by $\nabla^{2} \mathbf{V}$ (the minimum/maximum values of each field are given above each panel). While the velocity fields are smooth, the pressure fields require significant resolution. The challenge for lower-order discretizations such as FEM, Finite-Difference or Finite-Volume is to accurately reproduce both velocity and pressures for this problem.


Figure 3: Closeup of dynamic pressure field for this calculation on the sub-domain $-1 \leq x \leq 1$, $0 \leq z \leq 1$. The blocky nature is an artifact of the plotting procedure (and the blurring is due to some weird pdf compression issue I can't get rid of). As these are spectral solutions, the solution is actually continuous in $z$ and accurate up to the Nyquist frequency in $x$.
is a unitary matrix and

$$
E(z)=e^{-k z}\left[\begin{array}{cc}
(1-k z) & 0  \tag{25}\\
0 & 1
\end{array}\right] \quad \mathbf{c}=\left[\begin{array}{c}
-c_{2} \\
c_{3}
\end{array}\right]
$$

Substituting in the horizontal velocity boundary conditions at $z=0$ (and noting that $E(0)=I$ ) gives

$$
\begin{equation*}
\mathbf{c}=k Q \tilde{\mathbf{U}}_{0} \tag{26}
\end{equation*}
$$

(i.e. $c_{2}=-i \mathbf{k}^{T} \tilde{\mathbf{U}}_{0}$ ) and the full solution can be written as

$$
\begin{gather*}
\tilde{\mathbf{U}}(\mathbf{k}, z)=Q^{H} E Q \tilde{\mathbf{U}}_{0}  \tag{27}\\
\tilde{W}(\mathbf{k}, z)=-i \mathbf{k}^{T} \tilde{\mathbf{U}}_{0} z e^{-k z}  \tag{28}\\
\tilde{P}=-i 2 e^{-k z} \mathbf{k}^{T} \tilde{\mathbf{U}}_{0} \tag{29}
\end{gather*}
$$

with pressure gradient terms

$$
\begin{align*}
& \tilde{P}_{H}=2 e^{-k z} \mathbf{k} \mathbf{k}^{T} \tilde{\mathbf{U}}_{0}  \tag{30}\\
& \tilde{P}_{z}=i 2 e^{-k z} k \mathbf{k}^{T} \tilde{\mathbf{U}}_{0} \tag{31}
\end{align*}
$$

where the subscript $H$ refers to the horizontal components. For completeness, it is useful to write out Equation (27) explicitly as

$$
\left[\begin{array}{c}
\tilde{U}  \tag{32}\\
\tilde{V}
\end{array}\right]=\frac{e^{-k z}}{k^{2}}\left[\begin{array}{cc}
k_{x} & -k_{y} \\
k_{y} & k_{x}
\end{array}\right]\left[\begin{array}{cc}
(1-k z) & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{cc}
k_{x} & k_{y} \\
-k_{y} & k_{x}
\end{array}\right]\left[\begin{array}{c}
\tilde{U}_{0} \\
\tilde{V}_{0}
\end{array}\right]
$$

### 4.1 Special Case Ridge-Transform-Ridge solution

Equations (27)-(31) form a general solution for the flow of a viscous half-space driven by arbitrary horizontal flow at the surface and no flow as $z \rightarrow \infty .2$ For the specific case of flow near a mid-ocean ridge driven by two-rigid plates, we can orient the ridge such that the surface plate motion is always ridge perpendicular (i.e. parallel to the transforms). I.e. $\mathbf{U}(x, y)=$ $U(x, y) \mathbf{i}$ and therefore $V(x, y)=\tilde{V}_{0}=0$.

Under these conditions the solutions for the velocity, pressure and pressure gradients in the Fourier domain become

$$
\begin{align*}
\tilde{U} & =\tilde{U}_{0}\left(1-\frac{k_{x}^{2}}{k} z\right) e^{-k z}  \tag{33}\\
\tilde{V} & =-\tilde{U}_{0} \frac{k_{x} k_{y}}{k} z e^{-k z}  \tag{34}\\
\tilde{W} & =-i k_{x} \tilde{U}_{0} z e^{-k z}  \tag{35}\\
\tilde{P} & =-2 i k_{x} \tilde{U}_{0} e^{-k z} \tag{36}
\end{align*}
$$

With Pressure gradient terms

$$
\begin{align*}
\tilde{P}_{x} & =2 k_{x}^{2} \tilde{U}_{0} e^{-k z}  \tag{37}\\
\tilde{P}_{y} & =2 k_{x} k_{y} \tilde{U}_{0} e^{-k z}  \tag{38}\\
\tilde{P}_{z} & =i 2 k k_{x} \tilde{U}_{0} e^{-k z} \tag{39}
\end{align*}
$$

Which should be identical to the 2-D equations (15)-(17) for $k_{y}=0, k=\left|k_{x}\right|$.

### 4.2 Implementation

Figure 4 shows upper boundary conditions and pressure-velocity solutions for one solution of the 3-D equations for a simplified Ridge-Transform-Ridge calculation. These figures were produced by the example script seeRidge3D which calls setTopVelocity3D to set the upper velocity Boundary condition and specRidge3D to calculate the full spectral solution. Lifting a trick from Phipps Morgan and Forsyth [1988], the 3D solution uses superposition to solve for a uniform 2-D solution first ala specRidge2D, then adds in the flow due to non-2D spreading inherent in the upper Boundary conditions. Full details are buried in the codes.

## References

G. K. Batchelor. An Introduction to Fluid Dynamics. Cambridge Univ Press, Cambridge, 1967.
D. McKenzie. The generation and compaction of partially molten rock. J. Petrol., 25:713-765, 1984.
J. Phipps Morgan. Melt migration beneath mid-ocean spreading centers. Geophys. Res. Lett., 14:1238-1241, 1987.

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Figure 4: (a) Ridge-Transform-Ridge velocity upper Boundary conditions for the 3-D calculations. This solution is assumed periodic in $y$ with period $24 L$ and embedded in a large scale 2-D $x$-Periodic solution that spans $x \in[-24 L, 24 L]$ as in Figure 1 (b,c) 2 views of Pressure and velocity streamlines for example 3-D problem with upper boundary condition given in Fig. 4h but only shown on the 3-D domain $x \in[-6 L, 6 L], y \in[0,12 L], z \in[0,6 L]$ with $128 \times 128 \times 64$ points. Slices are colored by the Pressure field (figure produced by script seeRidge3D)
J. Phipps Morgan and D. W. Forsyth. 3-dimensional flow and temperature perturbations due to a transform offset - effects on oceanic crustal and upper mantle structure. J. Geophys. Res., 93:2955-2966, Apr 101988.
M. Spiegelman. Geochemical consequences of melt transport in 2-D: The sensitivity of trace elements to mantle dynamics. Earth Planet. Sci. Lett., 139:115-132, 1996.
M. Spiegelman, R. F. Katz, and G. Simpson. An introduction and tutorial to the "mckenzie equations" for magma migration. http://www.geodynamics.org/cig/workinggroups/magma/workarea/benchmark/McKenzieIntroBenchmarks.p 2006.
M. Spiegelman and D. McKenzie. Simple 2-D models for melt extraction at mid-ocean ridges and island arcs. Earth Planet. Sci. Lett., 83:137-152, 1987.


[^0]:    ${ }^{1}$ note for the 2-D solution $\mathbf{k}=\left[\begin{array}{ll}k_{x} & 0\end{array}\right]^{T}$ and $\left|\left|\mathbf{k} \|=\left|k_{x}\right|=|k|\right.\right.$.

[^1]:    ${ }^{2}$ If the Fourier Transforms are actually calculated by DFT's, the solution is also inherently doubly periodic in $x$ and $y$.

