2D non-Newtonian corner flow model of Mid-Ocean Ridges

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1 Model and flow geometry

I assume here corner flow under a perfectly horizontal plates (Fig. 1). The mantle is supposed incompressible, of uniform density ρ and obeys a powerlaw relation between stress and strain.

The model geometry suggests using a cylindrical reference frame, with coordinates r and θ, with θ defined as the angle from the vertical (Figure 1). Hence, the boundary conditions to verify are

$$\left\{ \begin{array}{lcl} v_r &=& V_0 \\ v_\theta &=& 0 \end{array} \right. \qquad \theta = \pm\pi/2 \tag{1}$$

with V_0 the half-spreading rate.

The incompressibility condition is verified by the introduction of the stream function ψ such that

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \tag{2}$$

$$v_\theta = -\frac{\partial\psi}{\partial r} \tag{3}$$

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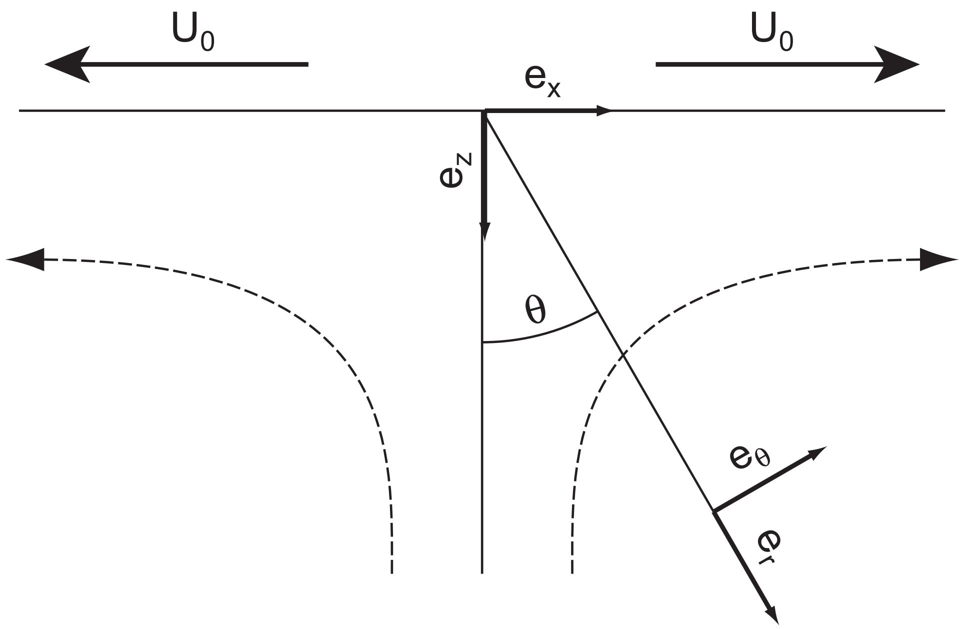

Figure 1: Configuration of the MOR model

As the boundary conditions do not depend on r, let us suppose that ψ depends on θ only. The solution is a similarity solution. It may also be recognized that ψ should be antisymmetric

$$\psi(-\theta) = -\psi(\theta) \tag{4}$$

The flow solution can be expressed as functions of ψ

$$\begin{array}{rcllcl} \psi & = & r\psi & & & \\ v_r & = & \psi' & & & \\ v_\theta & = & -\psi & & & \\ \dot{\varepsilon}_{rr} & = & \partial v_r/\partial r & & = & 0 \\ \dot{\varepsilon}_{\theta\theta} & = & \frac{1}{r}\left(v_r + \partial v_\theta/\partial\theta\right) & & = & 0 \\ \dot{\varepsilon}_{r\theta} & = & g/2r & & & \end{array} \tag{5}$$

where I define

$$g = \psi'' + \psi \tag{6}$$

For a powerlaw material, I define the effective viscosity as

$$\eta = B\dot{\varepsilon}_{\mathrm{II}}^{\frac{1-n}{n}}, \tag{7}$$

where $\dot{\varepsilon}_{II}$ is a strain rate invariant, which, for this geometry, is written simply as

$$\dot{\varepsilon}_{\mathrm{II}} = \left[\left(\frac{\dot{\varepsilon}_{xx} - \dot{\varepsilon}_{zz}}{2} \right)^2 + \dot{\varepsilon}_{xz} \right]^{1/2} = |\dot{\varepsilon}_{r\theta}| = \frac{|g|}{2r} \tag{8}$$

Then, the deviatoric stresses become

$$\begin{array}{lllll} \sigma_{rr} & = & \eta \dot{\varepsilon}_{rr} & = & 0 \\ \sigma_{\theta\theta} & = & \eta \dot{\varepsilon}_{\theta\theta} & = & 0 \\ \sigma_{r\theta} & = & \eta \dot{\varepsilon}_{r\theta} & = & B\,(2r)^{-1/n}\, g^{1/n} \end{array} \tag{9}$$

2 Stress equilibrium

The total pressure P can be decomposed into a lithostatic pressure $P_L = \rho g r \cos\theta$ and a non-lithostatic pressure

$$p = P - \rho g r \cos\theta \tag{10}$$

Using separation of variables, I define

$$p = R\,(r) \times \Theta\,(\theta) \tag{11}$$

Inserting this decomposition into the stress equilibrium equations, resolved in the r and θ direction, results in

$$\begin{array}{lllll} R'\,\Theta & = & \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} & = & \frac{1}{n} B\,(2r)^{-\frac{1-n}{n}}\, g' g^{\frac{1-n}{n}} \\ R\,\Theta' & = & \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial r} + 2\sigma_{r\theta} & = & \left(2 - \frac{1}{n}\right) B\,(2r)^{-\frac{1}{n}}\, g^{\frac{1}{n}} \end{array} \tag{12}$$

I use the first equation to define the r and θ dependence of the pressure field

$$R = B\,(2r)^{-1/n} \tag{13}$$

$$\Theta = -g' g^{\frac{1-n}{n}} = -n f' \tag{14}$$

where f is defined as

$$f = g^{1/n} \tag{15}$$

The second stress equilibrium equation further implies

$$f'' + l^2\, f = 0, \quad l^2 = \frac{2n-1}{n^2} \tag{16}$$

The flow underneath a mid-ocean ridge with a powerlaw rheology is therefore obtained by solving two ODEs, Eq. 16 and the following:

$$\psi'' + \psi = f^n \tag{17}$$

For a Newtonian fluid, $n = 1$, which implies $l = 1$. Then, these equation can be combined too form the classical biharmonic equation [Batchelor, 1967]

$$\nabla^4 \psi = 0 \tag{18}$$

3 Flow solution

The general solution of Eq. 16 is

$$f = f_1 \sin l\theta \ + f_2 \cos l\theta \tag{19}$$

Symmetry conditions at Mid-Ocean Ridges imply that $f_2 = 0$. The coefficient f_1 will be determined by matching the plate velocity with that of the solution.

For the time being, let's scale Eq. 17 by f_1^n. We obtain

$$T'' + T = (\sin l\theta)^n \,, \qquad \text{with } \psi = f_1^n T \tag{20}$$

As the tangential velocity $v_\theta = f_1^n \ T$ is null at the surface $\theta = \pi/2$ and at the symmetry axis $\theta = 0$, the boundary conditions on Eq. 20 are simply

$$\left\{ \begin{array}{ll} T' = f_1^{-n} V_0 \equiv 1/D, & \theta = \pi/2 \\ T = 0, & \theta = \pi/2 \end{array} \right. \tag{21}$$

Finding the value for D is part of the solution.

For the case $n = 1$, the solution corresponds to the well-known corner flow theory [Batchelor, 1967; McKenzie, 1969]

$$T = -\frac{1}{2}\theta \cos\theta \tag{22}$$

$$D = 4/\pi \tag{23}$$

For other values of n, a numerical solution is required. Tovish et al., [1978] give a series expansion of the solution for integer values of n.

I wrote a series of Matlab routines that solve Eq. 20 with the boundary conditions of Eq. 21 using a Finite Difference approach and a multigrid solver. This was developed as part of the class *12:521 Computational Geodynamics Modeling* that I have been teaching with Jian Lin in the MIT/WHOI Joint Program in Oceanography.

Figure 2 displays the functions ψ and ψ' for various values of n as well as the value of the D coefficient and the angle at which radial velocity changes sign (angle of corner) as functions of n.

Figure 3 compares the flow field for $n = 1$, $n = 3$, and $n = 10$, with or without lithostatic pressure.

The main program is `NN_corner`. This script requests two input: the power law exponent n, and a buoyancy number β. The later is defined as

$$\beta = \frac{\bar{\rho} g h}{n l B \left(D V_0 / 2h \right)^{1/n}} \tag{24}$$

It represent the relative strength of viscous vs. buoyancy forces. $\beta = 0$ ignores gravity. The total pressure is given by

$$P = n l B \left(D V_0 / 2h \right)^{1/n} \left[\beta r \cos\theta - \left(2r/h \right)^{-1/n} f' \right] \tag{25}$$

4 References

Batchelor, G. K., 1967/2000 *An introduction to fluid dynamics*, Cambridge University Press

McKenzie, D.P., 1969, Speculations on the consequences and causes of plate motions, *Geophysical Journal of the Royal Astronomical Society* **18**, 1-32

Tovish, A., G. Schubert, and D. P. Luyendyk, 1978, Mantle flow pressure and the angle of subduction: non-Newtonian corner flows, *Journal of Geophysical Research* **83**, 1238-1241

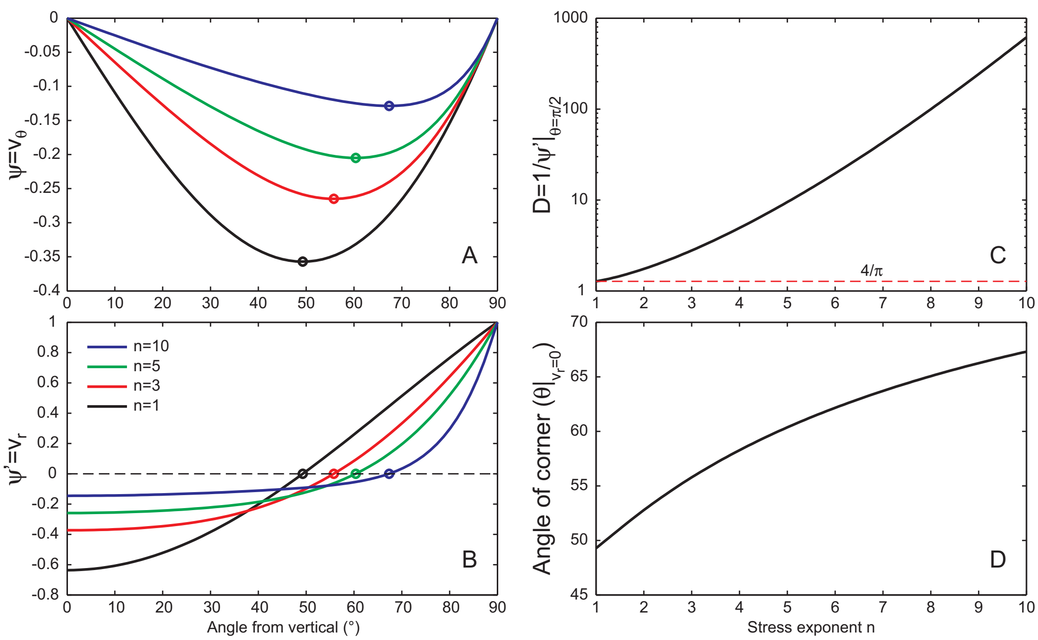

Figure 2: Numerical solution for (A) $\psi(\theta)$, (B) $d\psi/d\theta$ for n =1, 3, 5, and 10, (C) the coefficient D and (D) the angle at which the radial velocity is 0 as functions of the stress exponent n. In (A) and (B), the circles indicate the angle where $\psi' = 0$ (the corner of the corner flow). As n increases, the corner is shallower. In C, the line at $D = 4/\pi$ indicates the analytical solution for $n = 1$.

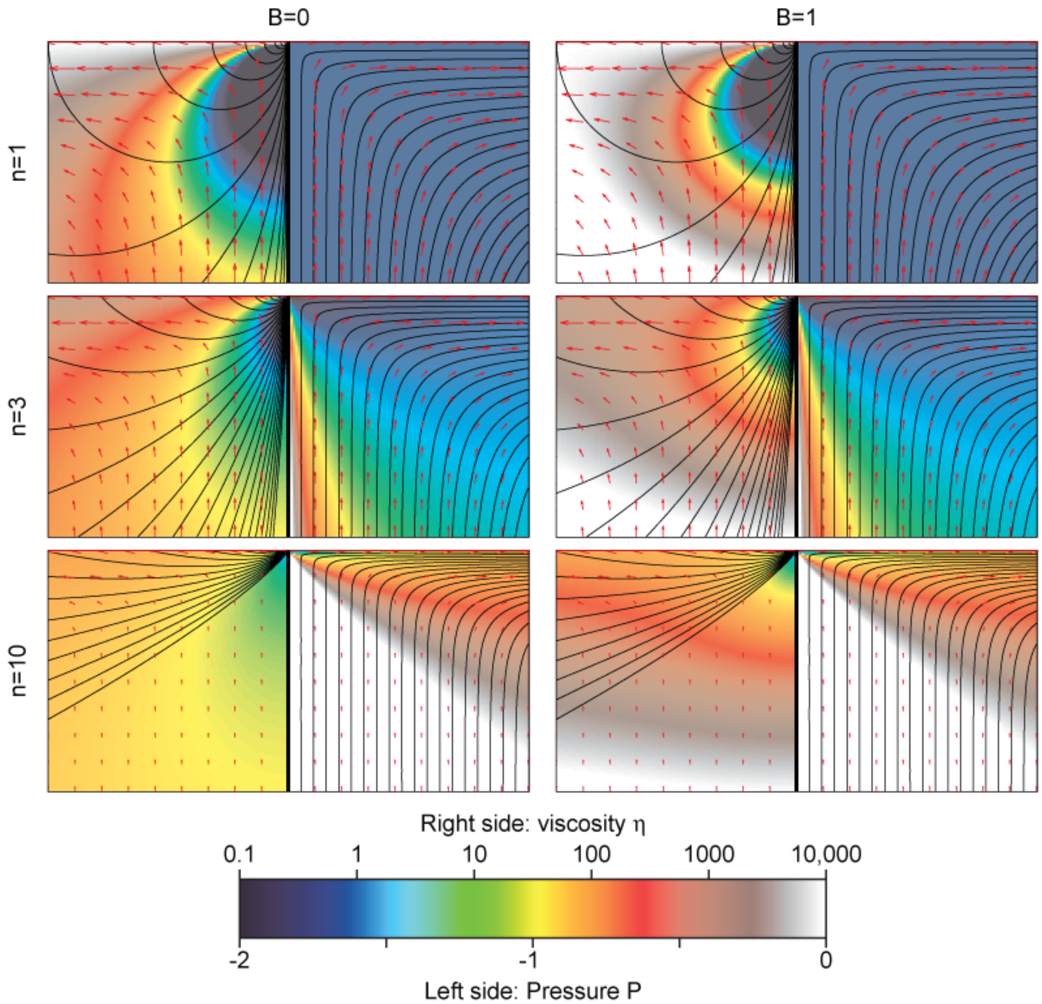

Figure 3: Visualization of flow field solution. Left column: $B = 0$ (no lithostatic pressure); Right column: $B = 1$; Top row: $n = 1$ (Newtonian); Middle row: $n = 3$; bottom row: $n = 10$. In each panel, the solution is mirrored underneath a ridge axis. Red arrows represent the velocity field. On the right-hand side, colors indicate the viscosity and contours represent mantle trajectories (contours of ψ). On the left-hand side, colors indicate the overpressure ($P < 0$ only, meaning suction; log-scale) and contours indicate the strain rate (contours of $\log_{10} \dot{\varepsilon}_{\mathrm{II}}$ from -4 to 2, every 1/4 log-units). The more non-linear the rheology, the shallower the corner and the lesser the suction term, because the upwelling region becomes almost rigid.