Inverse Problems in Geophysics: Examples and Remarks

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+several collaborators named during the talk.
Inverse problems and UQ

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How well do my data (and model) constrain the parameters? How uncertain do I have to be about the result?
Outline

Example I: Mantle convection & plate tectonics

Example II: Ice sheet boundary condition inversion

Common structure and challenges
Example 1: Mantle convection & plate tectonics

In cooperation with:

Michael Gurnis, Vishagan Ratnaswamy, Dunzhu Li (Caltech)
Introduction to mantle convection & plate tectonics

- Mantle convection is the thermal convection in the Earth’s upper ∼3000 km
- It controls the thermal and geological evolution of the Earth
- Solid rock in the mantle moves like viscous incompressible fluid on time scales of millions of years

- Driver for plate tectonics
- Stress release between plates causes earthquakes and tsunamis
- Subduction (left image) causes volcanic activity (chemical processes lead to rock melting)
- Theory of plate tectonics proposed by A. Wegener about 100 years ago; widely accepted in the 60s
Introduction to mantle convection & plate tectonics
Equations for momentum/mass/energy conservation

\[-\nabla \cdot \left[ \mu(T, u) \left( \nabla u + \nabla u^\top \right) \right] + \nabla p = f(T)\]

\[\nabla \cdot u = 0\]

\[\frac{\partial T}{\partial t} + u \cdot \nabla T - \nabla^2 T - \gamma = 0\]

\[\mathbf{u} \ldots \text{velocity}\]

\[p \ldots \text{pressure}\]

\[T \ldots \text{temperature}\]

\[\mu \ldots \text{viscosity}\]

**Rheology** is shear-thinning with plastic yielding, and upper/lower viscosity bounds; exponential dependence on temperature:

\[\mu(T, u) = \mu_{\text{min}} + \min \left( \frac{\tau_{\text{yield}}}{2\dot{\varepsilon}(u)}, w \min \left( \mu_{\text{max}}, a(T) \dot{\varepsilon}(u)^{\frac{1-n}{n}} \right) \right)\]

**Plates** are modeled as high-viscosity fluid (low \(T\)); plate boundaries are narrow zones of weak viscosity (prefactor \(\omega(x)\) controls plate coupling).
What we know (the data)

- Accurate present-day plate motion (from GPS)
- Historic plate motion for last few 100M yrs (from magnetic orientation in rocks, plants and animals)
- Stress normal to Earth’s surface (topography)
- State of stress between plates and for slabs/subducted plates (from earthquakes)
- Rock rheology from laboratory experiments (very different temperature/pressure/time scales)
- Blurry images of present-day Earth structure (by converting seismic wave speed to temperature)
What we would like to learn/infer (from data+models)

- Earth structure and history ("initial condition")
- Main drivers of plate motion: negative buoyancy forces or convective shear traction
- Energy dissipation in plate bending zones; strength of plate coupling
- Role of slab (=subducted plate) geometries
Inversion of plate coupling & rheology parameters

Bayesian inversion in instantaneous Stokes model

2D-box model

- Consistent plate motion (plate motion is an outcome)
- Shown are plate velocity data (top) and effective viscosity field (bottom)

Parameters and data

- Parameters: plate coupling strength, global rheology parameters ($\leq 5$ params)
- Data: plate velocities; currently extending to normal stress (topography data)
Inversion of plate coupling & rheology parameters

Bayesian inversion in instantaneous Stokes model

- Inference for plate coupling $\Gamma$ and rheology parameter $n$ from surface velocity data.
- Gaussian prior distribution for parameters.
- Posterior approximation uses inverse Hessian as covariance matrix.
- Full posterior computed using Delayed Rejection Adaptive Metropolis (DRAM) sampling.

2D pairwise conditionals for $\Gamma, n, \sigma_y$. Prior, Gaussian approximation and true posterior.
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2D pairwise marginals for $\Gamma, n, \sigma_y$.
Gaussian approximation and true posterior.
Inversion of initial temperature $T_0$ (and global parameters)

Forward simulation

- Use (estimate of) present-day mantle temperature and plate tectonic history to “go back” in time.
- Gradient computation requires solving state equation, and adjoint equation backwards in time.
- This is time- and memory-consuming.
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Iterates of deterministic inversion

- Data: Surface velocities and final temperature
- Parameters: Initial condition and global rheology parameters
- Method: Quasi-Newton method LBFGS, preconditioned
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- I’d like convergence to be faster
- Part of reason for slow convergence is flat valley in cost landscape indicating trade-off between parameters
Example II: Ice sheet boundary condition inversion

In cooperation with:

Omar Ghattas, Tobin Isaac (UT Austin), Noemi Petra (UC Merced)
Dynamics of the Antarctic ice sheet and sea level rise

Radar observations from a consortium of international satellites

Credit: NASA Goddard Space Flight Center/JPL-Caltech
Data → Inference → Prediction for Ice Sheets

"The uncertainty in the projections of the land ice contributions [to sea level rise] is dominated by the various uncertainties in the land ice models themselves ... rather than in the temperature projections." [From: IPCC Climate Change Report 2007]

Data: Current ice sheet geometry and surface ice flow velocity

Inference: for unknown basal boundary conditions and current state of the ice sheet under certain stationarity assumptions

Prediction: Sea level change by 2100 under different climate scenarios
Nonlinear Stokes ice sheet model

\[-\nabla \cdot \left[ 2\eta(u, n) \dot{\varepsilon}_u - Ip \right] = \rho g \text{ in } \Omega\]
\[\nabla \cdot u = 0 \text{ in } \Omega\]
\[\sigma_u n = 0 \text{ on } \Gamma_t\]
\[u \cdot n = 0, \ T\sigma_u n + \exp(\beta)Tu = 0 \text{ on } \Gamma_b\]

- $u$ ice flow velocity, $p$ pressure
- $\sigma_u = -Ip + 2\eta(u, n)\dot{\varepsilon}_u$ stress
- $\dot{\varepsilon}_u = \frac{1}{2}(\nabla u + \nabla u^T)$ strain rate
- $\eta(u, n) = \frac{1}{2} A^{-\frac{1}{n}} \dot{\varepsilon}^{\frac{1-n}{2n}}$ effective viscosity
- $\dot{\varepsilon}_II = \frac{1}{2} \text{tr}(\dot{\varepsilon}_u^2)$ second invariant of the strain rate tensor
- $\rho$ density, $g$ gravity
- $n$ unit normal vector
- $\beta$ log basal friction coefficient
- $T = I - n \otimes n$ tangential operator
- $\Gamma_t$ and $\Gamma_b$ top and base boundaries
Antarctic ice sheet inversion for basal friction field: InSAR data

Left: InSAR-based Antarctica ice surface velocity observations
Right: Inferred basal friction field (MAP point)
Antarctic ice sheet inversion for basal friction field: InSAR data

Left: Recovered ice surface velocity observations
Right: Inferred basal friction field (MAP point)
Antarctic ice sheet inversion for basal sliding field: InSAR data (Ronne ice shelf region)

Left: InSAR-based Antarctica ice surface velocity observations
Right: Inferred basal sliding field
Antarctic ice sheet inversion for basal sliding field: InSAR data (Ronne ice shelf region)

Left: Reconstructed ice surface velocity observations
Right: Inferred basal sliding field
Samples from prior distribution (top) and posterior approximation (bottom)
Variance of pointwise marginals

Prior and posterior (approximation) point marginals
Common structure and challenges
Common structure

\[ \min_p J(p) := \|u(p) - d\|^2 + R(p) \]

subject to

\[ \text{physics-model}(u, p) = 0 \]

We require 1st (and 2nd) derivatives of \( J \) with respect to \( p \):

- finite differences / forward sensitivities (requires number of parameter model solves)
- adjoint methods (requires single solve of adjoint equation)

- \( p \) ... parameters/cause
- \( d \) ... data
- \( R \) ... regularization / prior knowledge on \( p \)
Observations

- Formal inverse problem + adjoints is efficient (better: only) approach in high dimensions, hand-tuning parameters impossible (and subjective)
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- **Optimization** problem gives most likely parameters
- **Uncertainty quantification** powerful but computationally challenging: Derivative-based approximations or otherwise MCMC sampling
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- **Optimization** problem gives most likely parameters
- **Uncertainty quantification** powerful but computationally challenging: Derivative-based approximations or otherwise MCMC sampling
- Allows to **combine different data & models** (together with their uncertainties), e.g.:
  - current and historic plate velocities
  - normal stress,
  - slabs geometry,
  - heat loss, ...
Perspectives

Main challenges

- Availability of adjoints (derivation & implementation)

Remark: I believe that some CIG-related inverse/UQ problems can be considered BigData problems (and eligible for NSF funding under related programs).

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- Student training in optimization & statistics (beyond the general training in computational science, geophysics, HPC-codes,...)

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