Experience on Compressible Mantle Convection Code

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Compressible Mantle Convection Workshop
Outline

• Penalty method for incompressible flow
  ▪ Continuity and momentum equations
  ▪ Weak form
• Penalty method for compressible flow (failed attempts)
  ▪ Weak form I: asymmetric stiffness matrix
  ▪ Weak form II: symmetric stiffness matrix
• Mixed method for compressible flow
  ▪ Iterative algorithm
Incompressible flow

\[ u_{i,i} = 0 \]

\[ -p_{,i} + \sigma_{ij,j} = \Delta \rho g Ra \delta_{i2} \]

\[ \sigma_{ij} = \eta (u_{j,i} + u_{i,j}) \]
Penalty method

Let \( p = \lambda u_{j,j} \)

\[
\int w_ip_{,i}d\Omega = \int w_i(\lambda u_{j,j})_{,i}d\Omega
\]

\[
= \lambda \int n_iw_iu_{j,j}d\Gamma - \lambda \int w_{i,i}u_{j,j}d\Omega
\]

\[
= -\lambda \int w_{i,i}u_{j,j}d\Omega
\] symmetric
Compressible flow

\((\rho_r u_i)_i = 0\)

\(-p_i + \sigma_{ij,j} = \Delta \rho g Ra\delta_{i2}\)

\(\sigma_{ij} = \eta(u_{j,i} + u_{i,j} - \frac{2}{3}u_{k,k}\delta_{ij})\)
Penalty method I

Let \( p = \lambda(\rho_r u_j)_j \)

\[
\int w_i p, i d\Omega = \int w_i (\lambda(\rho_r u_j)_j)_i d\Omega \\
= \lambda \int n_i w_i (\rho_r u_j)_j d\Gamma - \lambda \int w_{i,i} (\rho_r u_j)_j d\Omega \\
= -\lambda \int (\rho_r w_{i,i} u_j)_j - \rho_{r,j} w_{i,i} u_j) d\Omega
\]

symmetric symmetric
Results

- Asymmetric stiffness matrix
  - solved by LAPACK solver
- Recovering $p$ is straightforward
- Benchmark:
  - stress field is inaccurate
Penalty method II

Let \( p_{i,j} = \lambda \rho_r (\rho_r u_j)_{,ji} \)

\[
\int w_i p_{i,j} \, d\Omega = \int w_i \lambda \rho_r (\rho_r u_j)_{,ji} \, d\Omega
\]

\[
= \lambda \int n_i w_i \rho_r (\rho_r u_j)_{,j} \, d\Gamma
\]

\[ - \lambda \int (\rho_r w_i)_{,i} (\rho_r u_j)_{,j} \, d\Omega \]

\[ = -\lambda \int (\rho_r w_i)_{,i} (\rho_r u_j)_{,j} \, d\Omega \]

\text{symmetric}
Results

- Symmetric stiffness matrix
  - solved by ConMan’s original solver
- Recovering $p$ is difficult
- Benchmark:
  - accurate in constant $\eta$ case
  - inaccurate in depth-dep. $\eta$ case
Iterative algorithm

\[ \nabla \cdot \mathbf{u} + \frac{1}{\rho_r} \frac{\partial \rho_r}{\partial z} u_z = 0 \]

\[
\begin{bmatrix}
K & G \\
G^T + C_\rho & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u} \\
p
\end{bmatrix} =
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

asymmetric matrix equation
Results

- Asymmetric stiffness matrix
  - solved by Uzawa algorithm using BiCGstab (very similar to the method of Dave and Luis)
- Good accuracy in $v$, $p$, $\sigma$
- Dissipative heating term
  - error amplified because of $A^2 - B^2$
  - element-level $\sigma$ suffers from pressure mode oscillation
  - node-level $\sigma$ (interpolated from element-level $\sigma$) is better
Lessons learnt

• Penalty method + non-iterative algorithm is difficult to be accurate for ConMan
  ▪ because of bilinear-velocity, constant-pressure element?
  ▪ using higher order element (e.g. enriched element [Fortin, 1981])?

• Mixed method + iterative algorithm is accurate for ConMan
  ▪ maybe will work for CitCom too