Two phase theory of compaction and damage

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Brittle-ductile behavior in lithosphere connects the pure brittle/frictional-sliding regime and the viscous/ductile regimes.
Damage 2: Grainsize reduction

- Mylonites indicate that grainsize reduction causes shear localization in lithosphere during creep such as through dynamic recrystalization.

- Fault gauge involves grainsize reduction by cataclastic processes
Two-Phase Damage Theory*

Basic Hypothesis:

- Cracks, fractures = voids ............ implies 2 phases:
  - “matrix” (host rock)
  - “fluid” (void-filling medium, e.g., water, or air)
- Deformational work goes into making voids or cracks
- Energy to make voids/cracks:
  $\approx$ surface energy on fracture surface
  $\approx$ surface energy on interface between phases
Approach (mild tutorial)

• Start with two *simple* viscous materials called **matrix** (= host) and **fluid** (= void filler)
  – Basic properties: densities ($\rho_m, \rho_f$), viscosities ($\mu_m, \mu_f$), etc.

• Mix them “simply” (isotropic, no phase changes)
Mixture’s additional properties

- Location of **fluid pores** and **matrix grains**:

  \[ \Theta = \begin{cases} 
  1, & \text{in pores} \\
  0, & \text{in grains} 
  \end{cases} \]

  such that fluid and matrix volumes within total volume \( \delta V \) are

  \[ \delta V_f = \int_{\delta V} \Theta dV, \quad \delta V_m = \int_{\delta V} (1 - \Theta) dV \]

- Location and orientation of **interface** \( \nabla \Theta \) and interface area:

  \[ \delta A_i = \int_{\delta V} |\nabla \Theta| dV \]

- Interfacial surface tension (energy): \( \gamma \)
Continuum theory

- Can’t track individual pores, grains and interfaces: use quantities that are **volume-averaged, continuous** (i.e., exist at all points):
  - Porosity (fluid volume fraction) $\phi = \frac{1}{\delta V} \int_{\delta V} \Theta dV$
  - Interface area per volume $\alpha(\phi) = \frac{\delta A}{\delta V} = A\phi^a (1 - \phi)^b$ where $A \sim (\text{grain/pore size})^{-1}$; $a, b \leq 1$ and interface curvature $\sim d\alpha/d\phi$

- Get governing equations in terms of averaged quantities, e.g., velocities

\[
\mathbf{v}_f = \frac{1}{\phi \delta V} \int_{\delta V} \mathbf{v}_f^{true} \Theta dV, \quad \mathbf{v}_m = \frac{1}{(1 - \phi) \delta V} \int_{\delta V} \mathbf{v}_m^{true} (1 - \Theta) dV
\]

- Until symmetry breaking assumption is made (regarding difference between phases), equations should be invariant to a switch of indices $f$ and $m$ (and $\phi$ with $1 - \phi$).
Mass conservation

- Growth in fluid volume governed by influx/efflux of fluid through surface exposure of pores on control volume; likewise for matrix volume:

- Result: equations for volume-fraction of pores and grains:

\[
\frac{\partial \phi}{\partial t} + \nabla \cdot [\phi \mathbf{v}_f] = 0 \quad \frac{\partial (1 - \phi)}{\partial t} + \nabla \cdot [(1 - \phi)\mathbf{v}_m] = 0
\]
Momentum conservation (force balance)

- Body force, e.g., gravity $g$, acts on pores and grains
- Fluid and matrix pressures $P_f, P_m$ and stresses $\tau_f, \tau_m$ act on surface exposures of pores and grains
- Interaction force: fluid surface forces (e.g., drag) acting on matrix through their interface and vice versa
Surface energy in two-phase theory

- Surface tension $\gamma$ acts as line force on intersection of interface with surface

- Surface energy exists at interface
  - Interface area per volume $\alpha = A\phi^a(1 - \phi)^b$ where $A \sim \frac{1}{\text{grain/pore-size}}$; $a, b \leq 1$ and $\phi$ is fluid volume fraction
  - Interface curvature $\sim \frac{d\alpha}{d\phi}$
Interaction (body) force

- Forces acting on fluid through interface (by matrix + interface)
- ... and on matrix through interface (by fluid + interface).
- Includes:
  - Common pressure force
  - Common viscous drag: $\pm c(v_m - v_f)$
    where $c \sim \frac{\text{viscosity}}{\text{permeability}}$
  - Interface surface tension
Resulting momentum equations

- **Fluid:**

\[
0 = -\phi \left[ \nabla P_f + \rho_f g \hat{z} \right] + \nabla \cdot \left[ \phi \tau_f \right] + c \Delta \mathbf{v} + \omega \left[ (P_m - P_f) \nabla \phi + \nabla (\gamma \alpha) \right]
\]

- **Matrix:**

\[
0 = -(1 - \phi) \left[ \nabla P_m + \rho_m g \hat{z} \right] + \nabla \cdot \left[ (1 - \phi) \tau_m \right] - c \Delta \mathbf{v} + (1 - \omega) \left[ (P_m - P_f) \nabla \phi + \nabla (\gamma \alpha) \right]
\]

where stress is \( \tau_j = \mu_j \left( \nabla \mathbf{v}_j + \left[ \nabla \mathbf{v}_j \right]^t - \frac{2}{3} (\nabla \cdot \mathbf{v}_j) \mathbf{I} \right) \) with \( j = f \) or \( m \).

- average and difference quantities are \( \bar{q} = \phi q_f + (1 - \phi) q_m \) and \( \Delta q = q_m - q_f \).

- \( \omega \) represents extent to which surface tension/energy is embedded in one phase or the other; for solid matrix and liquid fluid \( \omega \approx 0 \).
Energy Equations: Heating and Damage

- Consider all input and growth of energy in fluid and matrix, and on interface:

\[ \frac{\overline{D} T}{\rho c} \frac{\partial T}{\partial t} - T \frac{\overline{D}}{\partial t} \left( \alpha \frac{d \gamma}{dT} \right) - T \alpha \frac{d \gamma}{dT} \nabla \cdot \tilde{v} = Q - \nabla \cdot q + B \left( \frac{\overline{D} \phi}{\overpartial D \partial t} \right)^2 + (1 - f) \Psi \]

where “\( \tilde{\cdot} \)” means frame of reference of interface (i.e., \( \tilde{v} = \omega \mathbf{v}_f + (1 - \omega) \mathbf{v}_m \))
Interface Work and Damage

- **Equilibrium:**
  \[ P_m - P_f + \gamma \frac{d\alpha}{d\phi} = 0 \]

- **Quasi-equilibrium:**
  \[ P_m - P_f + \gamma \frac{d\alpha}{d\phi} = -B \frac{\widetilde{D}\phi}{Dt} \]

- **Far from equilibrium** (assume for now 1/grainsize $A$ is constant):
  \[ \left( P_m - P_f + \gamma \frac{d\alpha}{d\phi} \right) \frac{\widetilde{D}\phi}{Dt} = -B \left( \frac{\widetilde{D}\phi}{Dt} \right)^2 + f\Psi \]

where the deformational work is:

\[ \Psi = c\Delta v^2 + \phi \nabla \mathbf{v}_f : \mathbf{\tau}_f + (1 - \phi) \nabla \mathbf{v}_m : \mathbf{\tau}_m \]

Partitioning argument: $1 - f =$ fraction of deformational work going into dissipative heating. $f =$ remainder “stored” on interface, leads to damage.
• Micro-mechanical model:

\[ B = K \frac{\mu_m + \mu_f}{\phi(1 - \phi)} \]

where \( K \) is a dimensionless constant of \( O(1) \)
For $\gamma = 0$ and $\mu_f \ll \mu_m$ (hence $\omega \approx 0$)

$$\Delta P = \frac{K\mu_m}{\phi} \nabla \cdot \mathbf{v}_m$$

since in this case $\frac{\bar{D}\phi}{Dt} = \frac{D_m\phi}{Dt} = (1 - \phi) \nabla \cdot \mathbf{v}_m$

- Recovers the McKenzie 1984 momentum equations exactly, assuming a “bulk viscosity” of $K\mu_m/\phi$. 
Recent application: Source-sink driven 2D flow (BR2005)

- Velocity now given by

\[ \mathbf{v}_m = \text{“compressible” potential flow} + \text{toroidal flow} + \text{poloidal flow} \]

\[ \mathbf{v}_m = \nabla \theta + \nabla \times (\psi \mathbf{\hat{z}}) + \nabla \times \nabla \times (W \mathbf{\hat{z}}) \]

or

\[ \mathbf{v}_h = \nabla (\theta + \xi) + \nabla \times (\psi \mathbf{\hat{z}}) \quad \text{where} \quad \xi = \frac{\partial W}{\partial z} \]

- Source-sink \( S \) prescribes poloidal flow; vertical vorticity \( \Omega \) determines toroidal flow; dilation rate \( G \) determines “compressible” dilational/compactive potential:

\[ \nabla^2 \xi = S, \quad \nabla^2 \psi = -\Omega, \quad \nabla^2 \theta = G \]

- \( S \) imposed; but equations for \( \Omega \) and \( G \) are given by combined force and damage equations and are ugly:

\[ \nabla^2 \Omega = [\text{ugly mess}] \]

\[ \nabla^2 G = [\text{hideous mess}] \]
Void generating vs grain/void size reducing damage

• Recall interface area density defined as

\[ \alpha = A \eta(\phi) \text{ where } \eta(\phi) = \phi^a (1 - \phi)^b \]

and \( a, b \leq 1 \).

• \( A \) is effectively the inverse of the average grain and/or void size.

• If now we consider \( A \) as a variable, and allow damage to incur non-void interface growth then

\[ \gamma A \frac{d\eta}{d\phi} \frac{D\phi}{Dt} + \gamma \eta \frac{DA}{Dt} = -(P_m - P_f) \frac{D\phi}{Dt} - B \left( \frac{D\phi}{Dt} \right)^2 + f\Psi \]
• Assuming deformational work partitions between void and non-void interface growth, \( f = f_\phi + f_A \) we have

\[
\gamma A \frac{d\eta}{d\phi} \frac{\overline{D}_\phi}{Dt} = -(P_m - P_f) \frac{\overline{D}_\phi}{Dt} - B \left( \frac{\overline{D}_\phi}{Dt} \right)^2 + f_\phi \Psi \\
\gamma \eta \frac{\overline{D}_A}{Dt} = f_A \Psi
\]

• Also allow for grain-size dependent viscosity (using diffusion creep model for a general relation):

\[
\mu_m = \mu_0 (A_0/A)^m
\]
2-D source-sink flow tests...

- $S$ = driving source-sink flow field
- $G$ = void-generating dilational flow field
- $\Omega$ = toroidal (strike-slip) vorticity field
- $v_h$ = velocity
- $\phi$ = void fraction (porosity)
- $A$ = inverse grain/void-size
Void-generating damage: $f_A = 0$
time = 0.00535
\begin{align*}
S \ [\text{min/max} = -1/1] & \quad G \ [\text{min/max} = -0.0428/0.446] \\
\Omega \ [\text{min/max} = -0.00133/0.00558] & \quad V_h \ [\text{max vec.length} = 0.0596] \\
\phi \ [\text{min/max} = 0.04838, 0.05994] & \quad \text{time} = 0.0266
\end{align*}
$S$ [min/max = -1/1]

$G$ [min/max = -0.25/1.64]

$\Omega$ [min/max = -0.0117/0.0542]

$v_h$ [max vec.length = 0.147]

$\phi$ [min/max = 0.03538, 0.1188]

Time = 0.094
Void-generating damage generates strong dilational field that enhances apparent poloidal flow (even causes monopolar flow) and inhibits strike-slip/toroidal flow.
Fineness generating or grain size reducing damage: \( f_\phi = 0 \)
$S$ [min/max = -1/1]

$G$ [min/max = -0.0331/0.0337]

$\phi$ [min/max = 0.0485, 0.051]

$\Omega$ [min/max = -0.293/0.308]

$v_h$ [max vec.length = 0.0742]

$\alpha$ [min/max = 0.9914/1.119]

$\text{time} = 0.0232$
$S_{[\text{min/max} = -1/1]}$

$G_{[\text{min/max} = -0.00603/0.0118]}$

$\phi_{[\text{min/max} = 0.04351, 0.05745]}$

$\Omega_{[\text{min/max} = -0.721/0.875]}$

$v_h_{[\text{max vec.length} = 0.0941]}$

$\alpha_{[\text{min/max} = 0.993/1.284]}$

time = 0.544$
S [min/max = -1/1]

G [min/max = -0.0062/0.0124]

Φ [min/max = 0.0417, 0.05945]

Ω [min/max = -0.944/1.21]

V_h [max vec.length = 0.101]

α [min/max = 0.9934/1.307]

time = 0.863
Fineness-generating ("grainsize reducing") damage

- Fineness-generating (grain-reducing) damage does not involve (even suppresses) dilation and facilitates plate-like strike-slip/toroidal flow

- However, this treats only mean grainsize, not distribution of grainsizes and thus cannot treat healing (graingrowth/coarsening) simultaneously with damage