Overview of the High-Order ADER-DG Method for Numerical Seismology

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Elastic Wave Equations as Linear Hyperbolic System

The velocity-stress formulation of the wave equations can be written in the compact vector-matrix notation that provides the linear hyperbolic system

$$\frac{\partial Q_p}{\partial t} + A_{pq} \frac{\partial Q_q}{\partial x} + B_{pq} \frac{\partial Q_q}{\partial y} + C_{pq} \frac{\partial Q_q}{\partial z} = S_p$$

with the vector of unknowns and the usually sparse Jacobian matrices

$$Q_p = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \\ u \\ v \\ w \end{pmatrix}, \quad A_{pq} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + 2\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & 0 \end{pmatrix}$$
Elastic Wave Equations as Linear Hyperbolic System

The velocity-stress formulation of the wave equations can be written in the compact vector-matrix notation that provides the linear hyperbolic system

\[
\frac{\partial Q_p}{\partial t} + A_{pq} \frac{\partial Q_q}{\partial x} + B_{pq} \frac{\partial Q_q}{\partial y} + C_{pq} \frac{\partial Q_q}{\partial z} = S_p
\]

with the vector of unknowns and the usually sparse Jacobian matrices.

Multiplication of the governing equation with a test function \(\Phi_k\) and integration over a tetrahedral element \(T^{(m)}\) gives

\[
\int_{T^{(m)}} \Phi_k \frac{\partial Q_p}{\partial t} dV + \int_{T^{(m)}} \Phi_k \left( A_{pq} \frac{\partial Q_q}{\partial x} + B_{pq} \frac{\partial Q_q}{\partial y} + C_{pq} \frac{\partial Q_q}{\partial z} \right) dV = 0
\]

And integration by parts yields

\[
\int_{T^{(m)}} \Phi_k \frac{\partial Q_p}{\partial t} dV + \int_{\partial T^{(m)}} \Phi_k F_p^h dS - \int_{T^{(m)}} \left( \frac{\partial \Phi_k}{\partial x} A_{pq} Q_q + \frac{\partial \Phi_k}{\partial y} B_{pq} Q_q + \frac{\partial \Phi_k}{\partial z} C_{pq} Q_q \right) dV = 0,
\]

fluxes identical to the finite volume method.
Time Accuracy through the ADER Approach

(Toro et al., 2001; Toro & Titarev, 2002)

In the reference element the equation yields the relation for Arbitrary high order DERivatives:

\[
\frac{\partial^k Q_p}{\partial t^k} = (-1)^k \left( A^*_{pq} \frac{\partial}{\partial \xi} + B^*_{pq} \frac{\partial}{\partial \eta} + C^*_{pq} \frac{\partial}{\partial \zeta} \right)^k Q_q
\]

The time accuracy is obtained by replacing all time derivatives in the Taylor series expansion in time around \( Q_p \), at \( t_{\text{local}} = 0 \), by space derivatives

\[
Q_p(\xi, \eta, \zeta, t) = \sum_{k=0}^{N} \frac{t^k}{k!} \frac{\partial^k}{\partial t^k} Q_p(\xi, \eta, \zeta, 0)
\]

\[
Q_p(\xi, \eta, \zeta, t) = \sum_{k=0}^{N} \frac{t^k}{k!} (-1)^k \left( A^*_{pq} \frac{\partial}{\partial \xi} + B^*_{pq} \frac{\partial}{\partial \eta} + C^*_{pq} \frac{\partial}{\partial \zeta} \right)^k Q_q(\xi, \eta, \zeta, 0)
\]

This way, time integration can be computed analytically!

\[\Rightarrow \text{Order of time accuracy } = \text{ Order of space accuracy} \]
ADER-DG – A High-Order Scheme

- numerical convergence analysis
- high order convergence in space AND time
Anisotropic Material \((de \ la \ Puente \ et \ al. \ 2007)\)

Wave propagation includes **directional dependency**.

\[
\frac{\partial Q_p}{\partial t} + A_{pq} \frac{\partial Q_q}{\partial x} + B_{pq} \frac{\partial Q_q}{\partial y} + C_{pq} \frac{\partial Q_q}{\partial z} = S_p
\]

Therefore, the **whole elasticity tensor** has to be considered in Hooke’s law.

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{yz} \\
\sigma_{xz} \\
\sigma_{xy}
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & 2c_{14} & 2c_{15} & 2c_{16} \\
c_{12} & c_{22} & c_{23} & 2c_{24} & 2c_{25} & 2c_{26} \\
c_{13} & c_{23} & c_{33} & 2c_{34} & 2c_{35} & 2c_{36} \\
c_{14} & c_{24} & c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\
c_{15} & c_{25} & c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\
c_{16} & c_{26} & c_{36} & 2c_{46} & 2c_{56} & 2c_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{yz} \\
\varepsilon_{xz} \\
\varepsilon_{xy}
\end{pmatrix}
\]
Anisotropic Material

Wave propagation includes **directional dependency**.

\[
\frac{\partial Q_p}{\partial t} + \sum_{q} A_{pq} \frac{\partial Q_q}{\partial x} + \sum_{q} B_{pq} \frac{\partial Q_q}{\partial y} + \sum_{q} C_{pq} \frac{\partial Q_q}{\partial z} = S_p
\]

The physics of the **anisotropy effect** can be included by the **modified Jacobian** matrices:

\[
A_{pq} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -c_{11} & -c_{16} & -c_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{12} & -c_{26} & -c_{25} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{13} & -c_{36} & -c_{35} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{16} & -c_{66} & -c_{56} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{14} & -c_{46} & -c_{45} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{15} & -c_{56} & -c_{55} \\
-\frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Anisotropic Material
(example: Komatitsch et al. 2007)
Viscoelastic Material \citep{Kaiser2007}

Wave propagation includes \textbf{viscous attenuation} represented by two fundamental mechanical models: Hooke (springs), Stokes (dashpots).

• Combination of a spring and dashpot is called a \textbf{viscoelastic mechanism} and adds one equation of an \textbf{anelastic variable} for each stress component

• anelastic variables couple into the elastic system via a \textbf{reactive source term}

\[
\frac{\partial \mathbf{Q}_p}{\partial t} + \mathbf{A}_{pq} \frac{\partial \mathbf{Q}_q}{\partial x} + \mathbf{B}_{pq} \frac{\partial \mathbf{Q}_q}{\partial y} + \mathbf{C}_{pq} \frac{\partial \mathbf{Q}_q}{\partial z} = \mathbf{E}_{pq} \mathbf{Q}_q
\]

The physics of the \textbf{viscoelastic effect} can be included by the \textbf{enlarged} ($n_v = 9 + 6n$) \textbf{modified Jacobian} matrices and the matrix $\mathbf{E}$.

\[
\mathbf{E} = \begin{bmatrix}
 E_1 & \cdots & E_n \\
 \end{bmatrix} \in \mathbb{R}^{n_v \times n_v}
\]

\[
E_{\ell} = \begin{bmatrix}
 0 & \mathbf{E} \\
 0 & \mathbf{E}' \\
\end{bmatrix} \in \mathbb{R}^{n_v \times n_v}
\]

\[
\mathbf{E}' = \begin{bmatrix}
 E'_1 & 0 \\
 \vdots & \ddots \\
 0 & E'_n \\
\end{bmatrix} \in \mathbb{R}^{6n \times 6n}
\]

\[
\mathbf{E}_{\ell} = \begin{bmatrix}
 -\lambda Y^\ell_{\ell} - 2\mu Y^\mu_{\ell} & -\lambda Y^\ell_{\ell} & -\lambda Y^\ell_{\ell} & 0 & 0 & 0 \\
 -\lambda Y^\ell_{\ell} & -\lambda Y^\ell_{\ell} - 2\mu Y^\mu_{\ell} & -\lambda Y^\ell_{\ell} & 0 & 0 & 0 \\
 -\lambda Y^\ell_{\ell} & -\lambda Y^\ell_{\ell} & -\lambda Y^\ell_{\ell} - 2\mu Y^\mu_{\ell} & 0 & 0 & 0 \\
 0 & 0 & 0 & -2\mu Y^\mu_{\ell} & 0 & 0 \\
 0 & 0 & 0 & 0 & -2\mu Y^\mu_{\ell} & 0 \\
 0 & 0 & 0 & 0 & 0 & -2\mu Y^\mu_{\ell} \\
\end{bmatrix}
\]
Viscoelastic Material (example: LOH.3, Day et al. 2003)
Poroelastic Material \((\text{de la Puente et al. 2007})\)

Wave propagation includes pores in a solid which are filled with a viscous fluid.

- A new wave type appears: the slow P-wave depending on \((K, \rho, T)\)

\[
\frac{\partial Q_p}{\partial t} + A_{pq} \frac{\partial Q_q}{\partial x} + B_{pq} \frac{\partial Q_q}{\partial y} + C_{pq} \frac{\partial Q_q}{\partial z} = E_{pq} Q_q
\]

\[
Q = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, u, v, w, p, u_f, v_f, w_f)^T
\]

\[
A_{pq} = -\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & c_{11}^u & c_{16}^u & c_{15}^u & 0 & \alpha_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{12}^u & c_{26}^u & c_{25}^u & 0 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{13}^u & c_{36}^u & c_{35}^u & 0 & \alpha_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{16}^u & c_{66}^u & c_{56}^u & 0 & \alpha_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{14}^u & c_{46}^u & c_{45}^u & 0 & \alpha_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{15}^u & c_{56}^u & c_{55}^u & 0 & \alpha_5 & 0 & 0 \\
1/\rho_x^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_x^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_x^{(2)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Homogeneous Poroelastic Material

Wave propagation includes pores in a solid which are filled with a viscous fluid.

- A new wave type appears: the slow P-wave depending on \((K_f, \rho_f, T)\)

\[
\frac{\partial Q_p}{\partial t} + A_{pq} \frac{\partial Q_q}{\partial x} + B_{pq} \frac{\partial Q_q}{\partial y} + C_{pq} \frac{\partial Q_q}{\partial z} = E_{pq} Q_q
\]

\[
E = 0.00070898
\]
Heterogeneous Poroelastic Material
Heterogeneous Poroelastic Material

Comparison with FD results provided by T. Müller and F. Krzikalla
\( p_\tau \) Adaptivity \( \text{(Dumbser et al., 2007)} \)

- Adapting the order ranging from \( O_4 \) to \( O_7 \) according to the *insphere radius*, which is responsible for *timestep restriction*

\[ \rightarrow 2 \times \text{larger time step w.r.t. pure } O_7 \]

\[ \rightarrow \text{only } 28\% \text{ of number of degrees of freedom w.r.t. pure } O_7: 400,000 \text{ instead of } 1.4 \text{ million} \]

\[ \rightarrow 6 \times \text{faster than pure } O_7 \text{ simulation, but comparable resolution} \]
Each tetrahedral element \( (m) \) has its own time step

\[
\Delta t^{(m)} < \frac{1}{2N + 1} \cdot \frac{l_{min}^{(m)}}{a_{max}^{(m)}},
\]

where \( l_{min} \) is the insphere radius of the tetrahedron and \( a_{max} \) is the fastest wave speed.

Therefore, the Taylor series in time depends on the local time level \( t^{(m)} \)

\[
Q_p(\xi, \eta, \zeta, t) = \sum_{k=0}^{N^{(m)}} \frac{(t - t^{(m)})^k}{k!} \frac{\partial^k}{\partial t^k} Q_p(\xi, \eta, \zeta, t^{(m)}),
\]
Comparison of computational effort (element updates) for *global* and *local* time stepping schemes:
Number of element updates:
- $72 \times 10^9$ for O6 with global time stepping
- $95 \times 10^7$ for O6 with local time stepping

Speed-up-factor: 100!
Complex External Source Terms (Käser, Mai, Gallovic, Dumbser, 2007)

\[
\frac{\partial Q_p}{\partial t} + A_{pq} \frac{\partial Q_q}{\partial x} + B_{pq} \frac{\partial Q_q}{\partial y} + C_{pq} \frac{\partial Q_q}{\partial z} = S_p
\]

- Point sources (Dirac in space) can be integrated **exactly** in an element

\[I_p^{(m)} = \int_{t^n}^{t^{n+1}} \int_{T^{(m)}} \Phi_k S_p(\vec{x}, t) \, dV \, dt\]

\[I_p^{(m)} = \sum_{i=1}^{n_s} \Phi_k(\vec{\xi}_S^{(i)}) \int_{t^n}^{t^{n+1}} M_p^{(i)}(t) \, dt\]

\[
M_{pq}^{DC} = \begin{pmatrix}
M_{xx} & M_{xy} & M_{xz} \\
M_{xy} & M_{yy} & M_{yz} \\
M_{xz} & M_{yz} & M_{zz}
\end{pmatrix}
= \begin{pmatrix}
M_1(t) & M_4(t) & M_6(t) \\
M_4(t) & M_2(t) & M_5(t) \\
M_6(t) & M_5(t) & M_3(t)
\end{pmatrix}
\]

- **Location** and **shape** of rupture fault is arbitrary (point cloud representation)
- **High-order polynomial approximation** in space allow for very coarse grids
Complex External Source Terms

- Time [s]
  - u [m/s]
  - v [m/s]

- Time [s]
  - V [m/s]
Complex External Source Terms
Mesh Generation for Complex 3D Geometries

- **Discontinuous Galerkin Finite Element Method** achieves arbitrary high approximation order in **space** and **time**
- **highly-developed automatic mesh generation** of unstructured tetrahedral meshes (can be problematic for hexahedral elements)
- **topography** of the free surface (precise digital elevation models)
- enormous amount of elements for **realistic 3D applications**
- optimized **mesh partitioning** (graph theory)
Mesh Generation for Complex 3D Geometries

Usage of **commercial tools** for
- geometry creation and modification
- mesh generation and optimization

e.g. GoCAD, ICEM CFD, GAMBIT
Modeling of Scattered Waves in Merapi Volcano

(J. Wassermann)

- **Problem adapted** mesh generation
- **p-adaptive** calculations to resolve topography very accurately
- **Load balancing** by grouping subdomains
Modeling of Scattered Waves in Merapi Volcano
(J. Wassermann)

• strong scattering effect of surface topography!
ADER-DG → what is currently done

**Topic**

- profound, quantitative accuracy analysis (Hermann)
- ADER-DG on quadrilateral and hexahedral meshes (Castro)
- locally implicit time stepping schemes (Dumbser)
- improving performance analysis and dynamic load balancing (Rivera, Brehm)
- fault properties and their seismic signature (Gallovic, Burjanek)
- volcano seismology (Wassermann)
- rotational seismology (Pham, Igel)

ADER-DG → what should/could be done

**Topic**

- combination of mesh topologies → tetrahedral / hexahedral
- improving memory/cache behaviour using space filling curves
- near surface scattering → effects on rotations
- dynamic rupture simulation
- reservoir modeling
What was missing? Better weather for Mt. Washington ... yesterday!