Implementing the compressible Anelastic approximation in a mantle convection code

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Technical overview

• Experience with spectral 3-D spherical code (direct solver) from Gary Glatzmaier helped, but details quite different:
  • Primitive variables (velocity & pressure)
  • Velocity components (3) & pressure staggered so that finite-difference derivatives always involve adjacent components
  • 3-D variable viscosity and (potentially) other properties
The Anelastic approximation: A straightforward step from the Boussinesq approximation

• **Boussinesq:** $\nabla \cdot \vec{v} = 0$
  - $\rho$ constant except in buoyancy term

• **Anelastic** $\nabla \cdot (\bar{\rho} \vec{v}) = 0$
  - $\rho = \bar{\rho}(z)$ except in buoyancy term
  - Viscous dissipation & adiabatic heating terms added to energy equation
  - $\text{Div}(\vec{v})$ term added in normal stresses
  - Properties like expansivity, conductivity can vary (as they can in the Boussinesq approx.)
Implementation: Continuity

- Finite volume discretized equation requires simple modification: e.g., in 2-D

**Boussinesq:**

\[
\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta z} = 0
\]

**Anelastic:**

\[
\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{\bar{\rho}_{j+1/2}v_{i,j+1/2} - \bar{\rho}_{j-1/2}v_{i,j-1/2}}{\Delta z} = 0
\]

Where \(i=x\) index, \(j=y\) index, \(u,v\) are \(x\) & \(y\) velocities, deltas are grid spacings, velocities are staggered so that they lie at desired locations.
Implementation: Energy equation

- Viscous dissipation and adiabatic heating/cooling terms can be treated explicitly (wrt time) and do not affect the stability of the numerical solution scheme, just as diffusion and advection can be treated explicitly.

- Calculated using standard finite difference/finite volume discretization
Implementation: Momentum eqn.

• Normal stresses gain a \( \text{div}(\mathbf{v}) \) term:

\[
\tau_{ij} = 2\eta \left( \dot{e}_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \right) = 2\eta \left( \dot{e}_{ij} - \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right)
\]

• If simply implemented this can destabilize iterations towards a velocity-pressure solution!
  – Because \( \text{div}(\mathbf{v}) \) is inaccurate during iterations
  – …and we take the divergence of this term!

• Fix: Use continuity equation to re-express this in terms of vertical velocity (next slide)
Re-express $\text{div}(\mathbf{v})$ in stress eq.

Continuity: \[ \nabla \cdot (\rho \mathbf{v}) = 0 = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho \]

Hence: \[ \nabla \cdot \mathbf{v} = -\frac{\mathbf{v} \cdot \nabla \rho}{\rho} \]

If $\bar{\rho}(z)$
\[ \nabla \cdot \mathbf{v} = -v_z \frac{\partial \bar{\rho}}{\partial z} \frac{1}{\bar{\rho}} = -v_z \frac{\partial \ln \bar{\rho}}{\partial z} \]

Finally:
\[ \tau_{ij} = 2\eta \left( \dot{e}_{ij} + \frac{1}{3} \delta_{ij} v_z \frac{\partial \ln \bar{\rho}}{\partial z} \right) \]

- Iterations converge stably with this form
- This was the only major problem in implementation!
Reference state, i.e., properties(z)

• In Stag3d, constructed self-consistently using some simple thermodynamic relations, typically from papers that were recent in the early 1990s (details in 1996 JGR).

• Problem arises in fitting upper mantle: real profile of density (& probably other properties) is strongly influenced by complex multi-component phase changes. Also, temperature-dependence is important for thermal conductivity, introducing lateral variations.

• Perhaps best to either
  – Fit PREM, or better:
  – Calculate density etc. as a function of T, p and “composition” (e.g., 5 main oxides) using a full thermodynamic database approach, minimizing free energy,… (so far implemented by Connelly, Matas & Ricard)
Enhancements: Phase changes introduce complexity

- 2-component (olivine, pyroxene) each with its own phase changes: A different reference state for each phase (4 olivine, 5 pyroxene).
- Reference density depends on phase => lateral as well as vertical variations.
- Advecting potential temperature rather than actual temperature automatically accounts for adiabatic heating/cooling and phase change latent heat.
Conclusions

• Changing from Boussinesq to compressible anelastic was straightforward, the main problem being finding a stable treatment of \( \text{div}(v) \) in the stress terms in the momentum equations

• Many additional stages of complexity/realism are possible so approaches should be flexible.