# Problems in magma dynamics

(and a few solutions)

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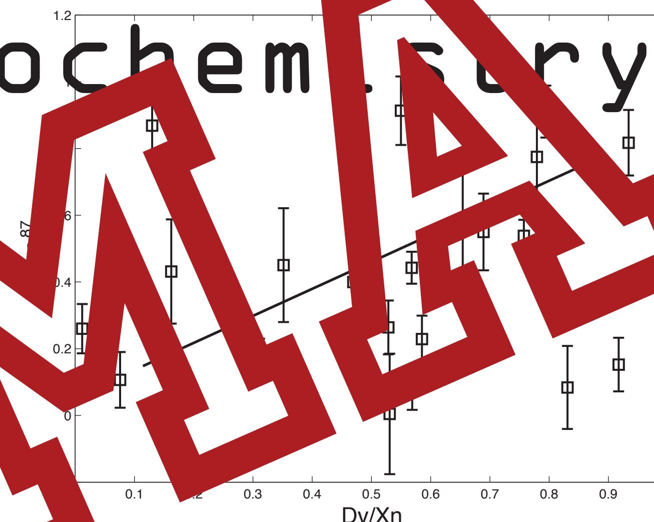
# Magma dynamics: why care!?

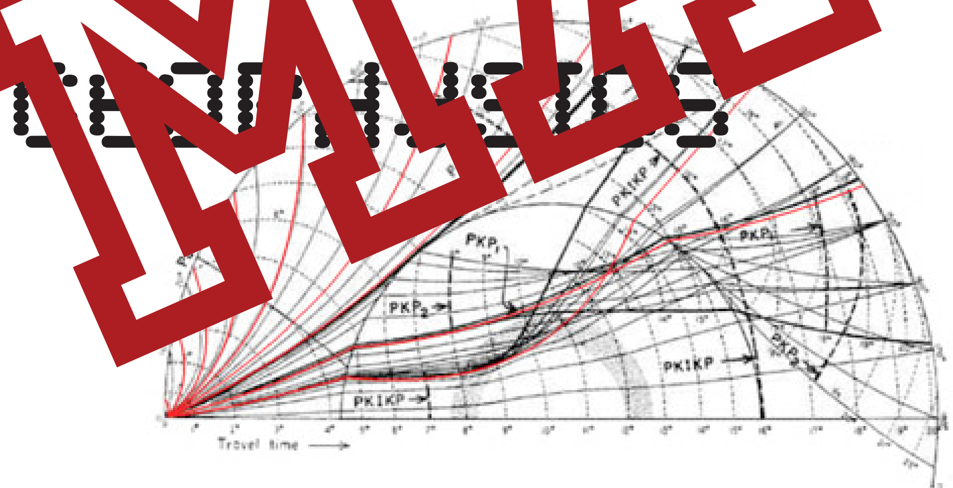
Reason 1) You love sets of coupled, non-linear PDEs

Reason 2)

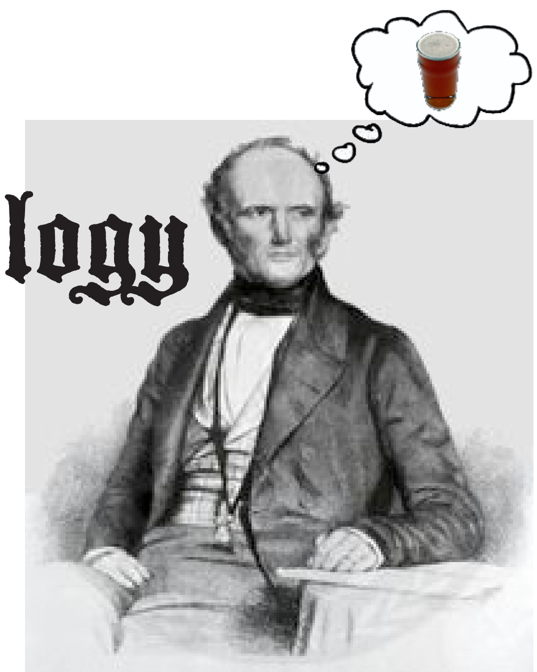

Geochemistry




MAGMA

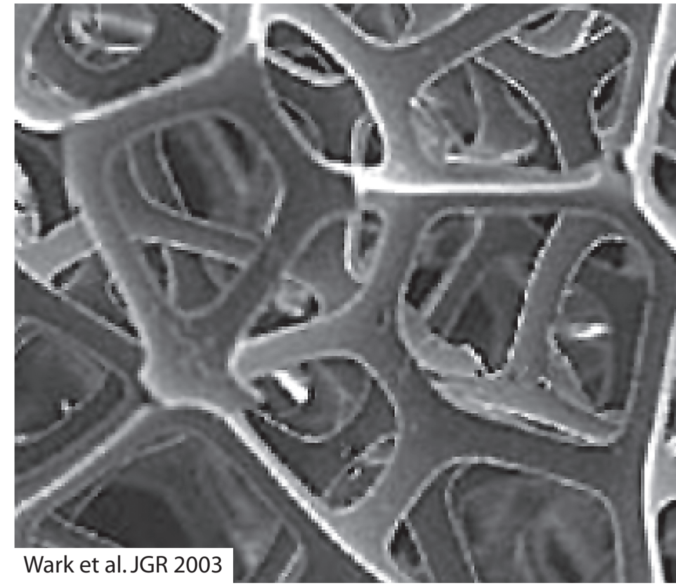



Geology



# Problem 1: Compaction is king

- 2 interpenetrating fluids w/ different viscosity, density, etc.
- Average over many grain/pores.
- Matrix: shear flow, compactive flow. Magma: buoyancy, porous flow.
- Lim zero porosity = Stokes mantle convection.




- Lim no compaction = Darcy porous flow.

| | | | |
|---|---|---|---|
| $\phi$ | Porosity | $\Gamma$ | Melting rate |
| $\mathbf{v}_f$ | Fluid velocity | $\eta, \xi, \mu$ | Shear, bulk, fluid viscosity |
| $\mathbf{v}_m$ | Matrix velocity | $K$ | Permeability |
| $P$ | Fluid pressure | $\rho$ | Density |

# Problem 1: Compaction is king

e.g. McKenzie (1984)

$$\frac{\partial \rho_f \phi}{\partial t} + \nabla \cdot [\rho_f \phi \mathbf{v}_f] = \Gamma \quad \textit{Fluid}$$

$$\frac{\partial \rho_m (1-\phi)}{\partial t} + \nabla \cdot [\rho_m (1-\phi) \mathbf{v}_m] = -\Gamma \quad \textit{Solid}$$

*Conservation of mass*

Make extended Boussinesq approximation.

Permeability is a non-linear function of porosity!

$$K \propto \phi^n$$

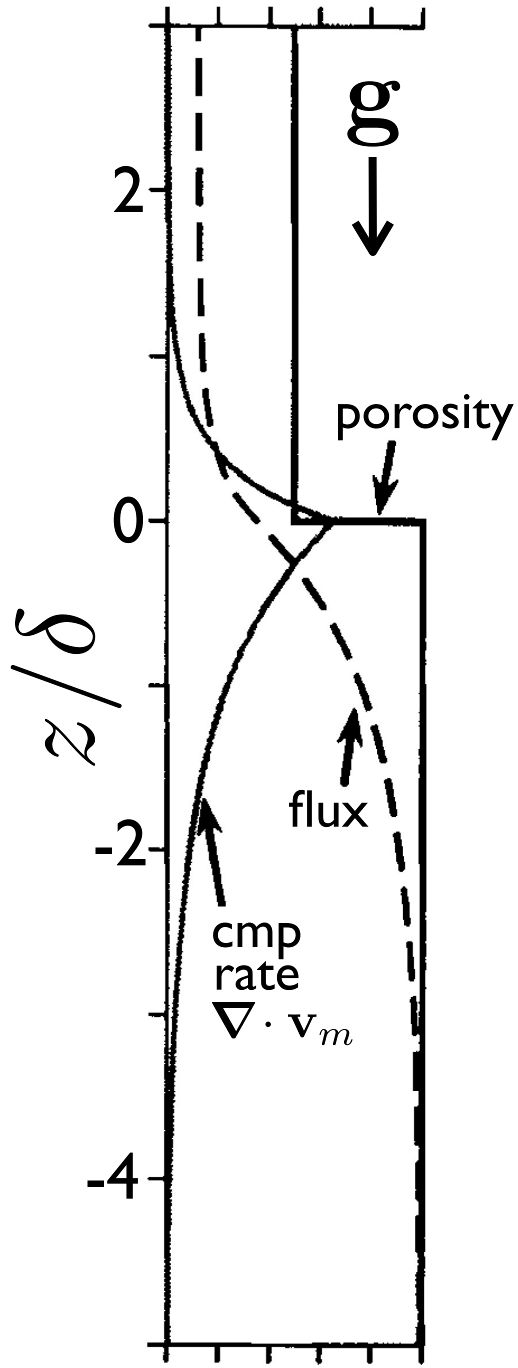
$$\phi(\mathbf{v}_f - \mathbf{v}_m) = -\frac{K}{\mu}[\nabla P - \rho_f g] \quad \textit{Fluid}$$

$$\nabla P = \nabla \cdot [\eta(\nabla \mathbf{v}_m + \nabla \mathbf{v}_m^T)] + \nabla [\xi \nabla \cdot \mathbf{v}_m] + \bar{\rho} g \quad \textit{Solid}$$

*Conservation of momentum*

Stresses arising from gradients in compaction!

# Problem 1: Compaction is king



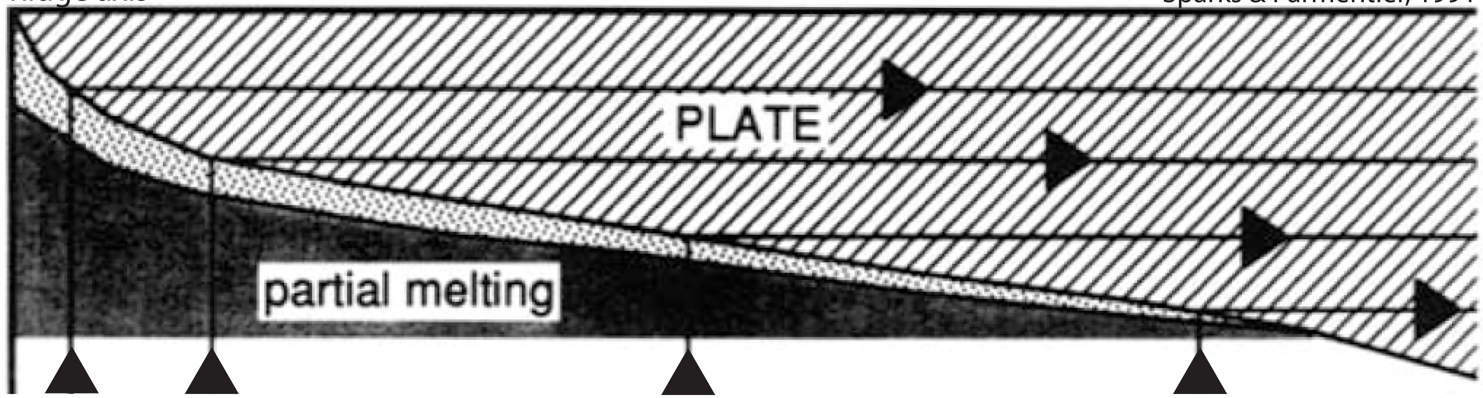

**Natural length-scale:
Compaction length**

$$\delta = \sqrt{\frac{(\zeta + 4\eta/3)K}{\mu}}$$

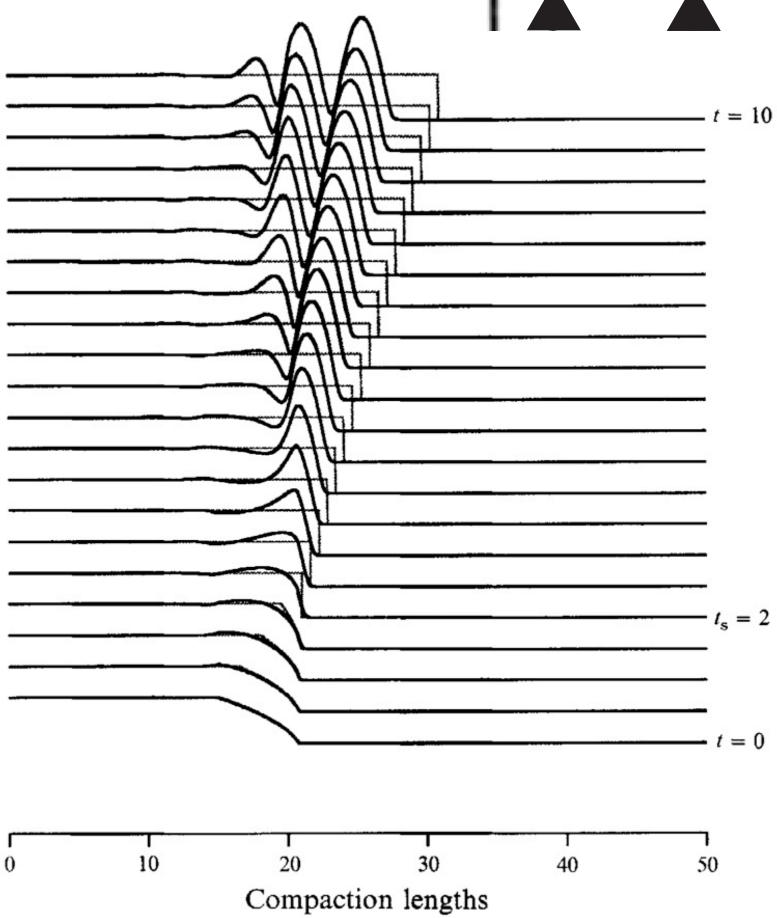
matrix bulk viscosity $\zeta$
matrix shear viscosity $\eta$
permeability $K$
melt viscosity $\mu$

Spiegelman, JFM 1993b

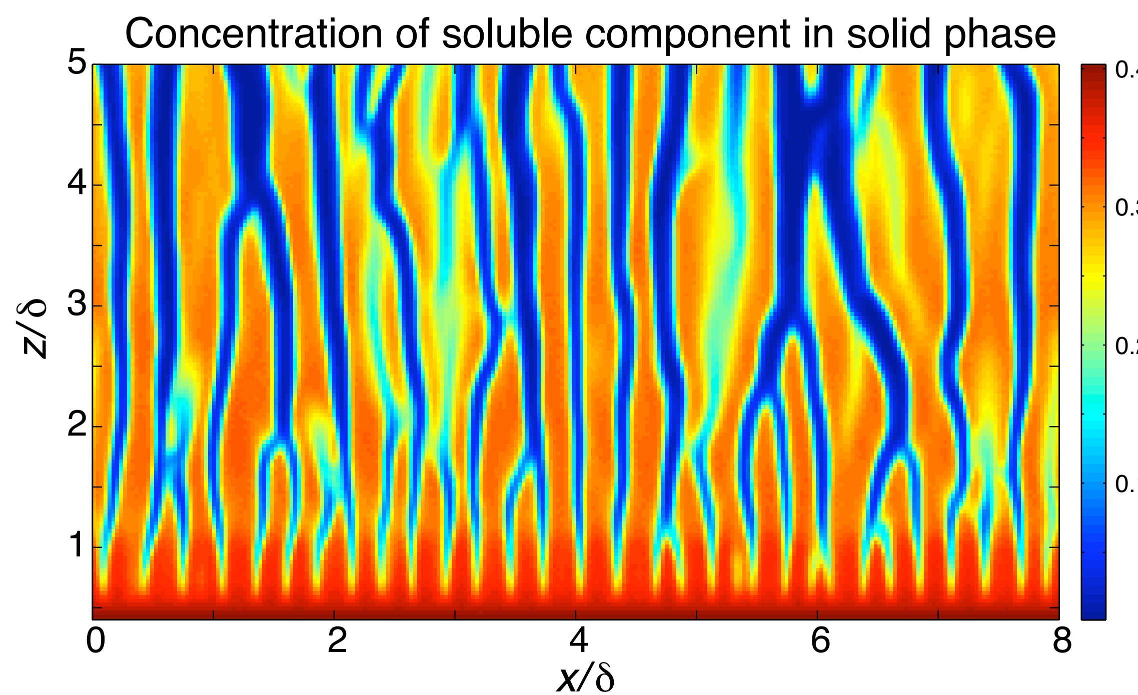
# Problem 1: Compaction is king



Sparks & Parmentier, 1991

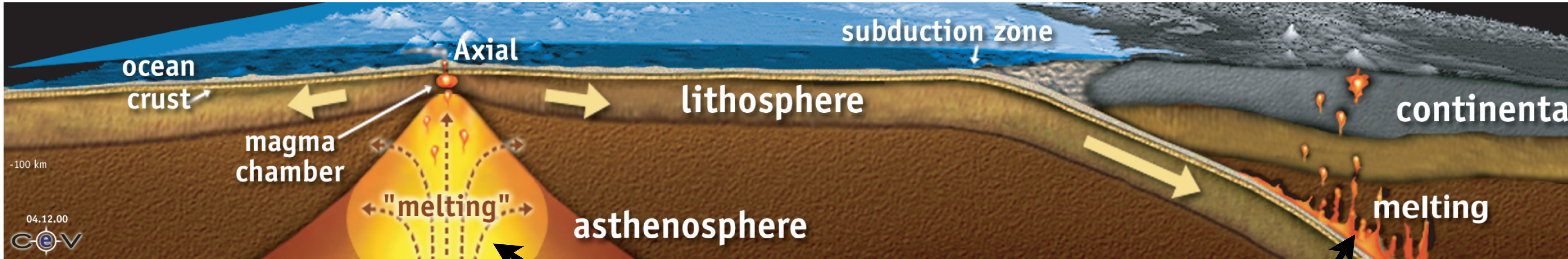


Spiegelman, JFM 1993b



after Spiegelman *et al.* 2001

# Problem 2: Phi goes to zero




Most other places, melt fraction is zero.

Where partial melting occurs, melt fraction is non-zero.

# Problem 2: Phi goes to zero

cmp viscosity vs. porosity: $\xi \mathrel{\widetilde{\propto}} \frac{1}{\phi}$ singularity!

Pressure decomposition: $P = \rho g z + \mathcal{P} + p$

Fluid pressure Lithostatic Compaction "Dynamic"

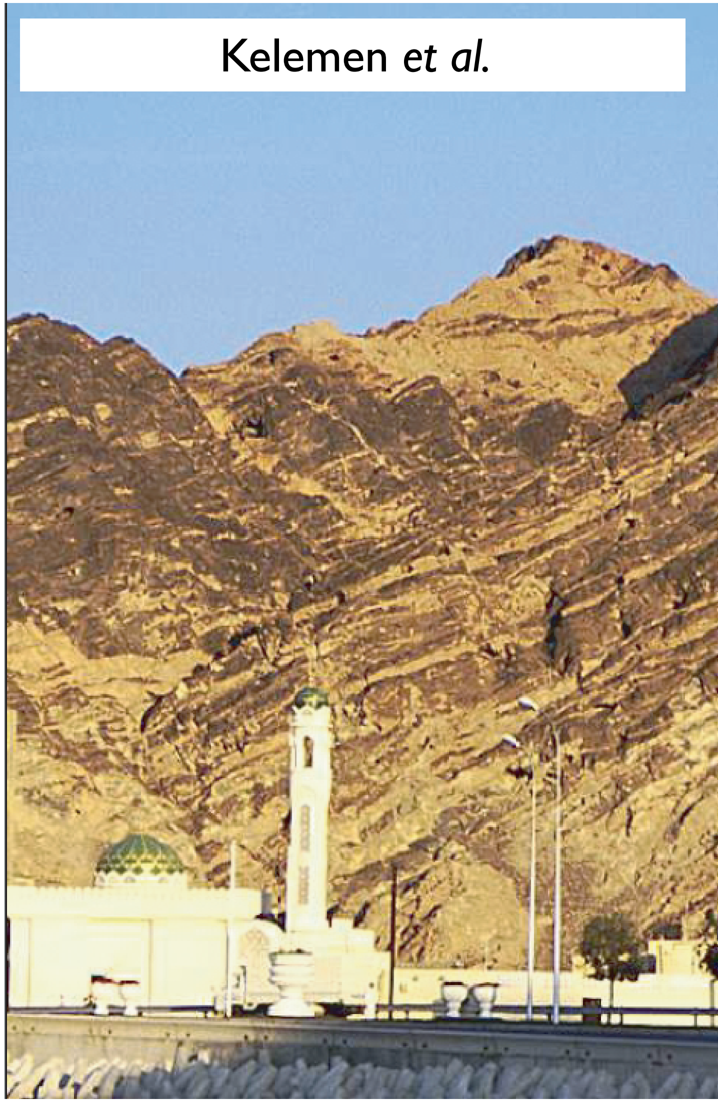
$$\boldsymbol{\nabla} \cdot \mathbf{v}_m = \mathcal{P}/\xi$$

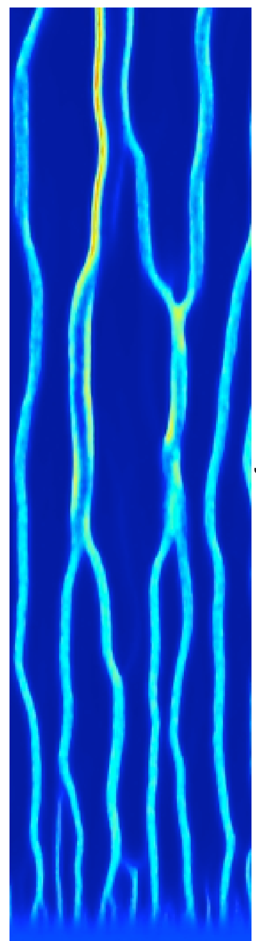
$$\boldsymbol{\nabla} p = \boldsymbol{\nabla} \cdot [\eta(\boldsymbol{\nabla}\mathbf{v}_m + \boldsymbol{\nabla}\mathbf{v}_m^T)] + \phi \Delta \rho g$$

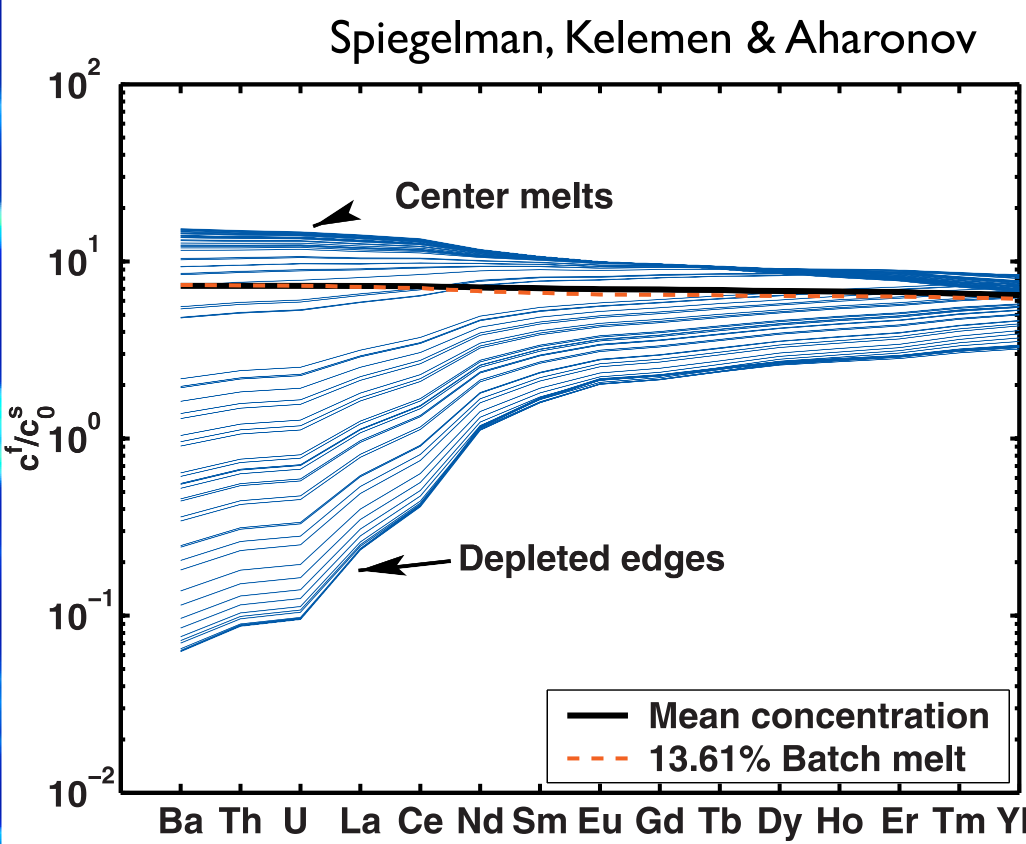
$$-\boldsymbol{\nabla} \cdot \frac{K}{\mu} \boldsymbol{\nabla} \mathcal{P} + \frac{\mathcal{P}}{\xi} = \boldsymbol{\nabla} \cdot \frac{K}{\mu}[\boldsymbol{\nabla} p + \Delta \rho g] + \Gamma \frac{\Delta \rho}{\rho^2}$$

Katz *et al.* 2007

# Problem 3: Think global, melts localize





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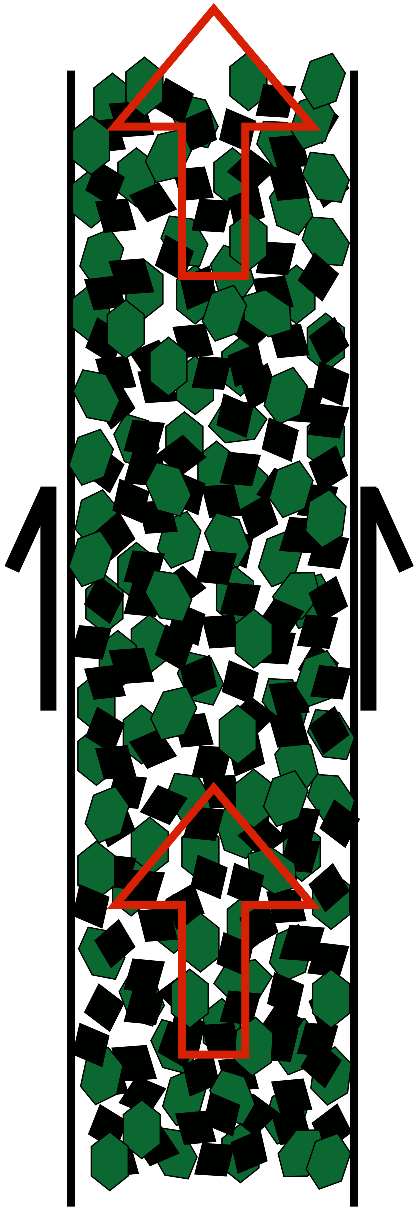


Olivine: $(Mg,Fe)SiO_2$

Pyroxene: $(Mg,Fe)Si_2O_6$

Mantle flow

Magmatic flow

Adiabatic (decompression) melting

$$\Gamma \propto (1-\phi)W\frac{dF}{dz}$$

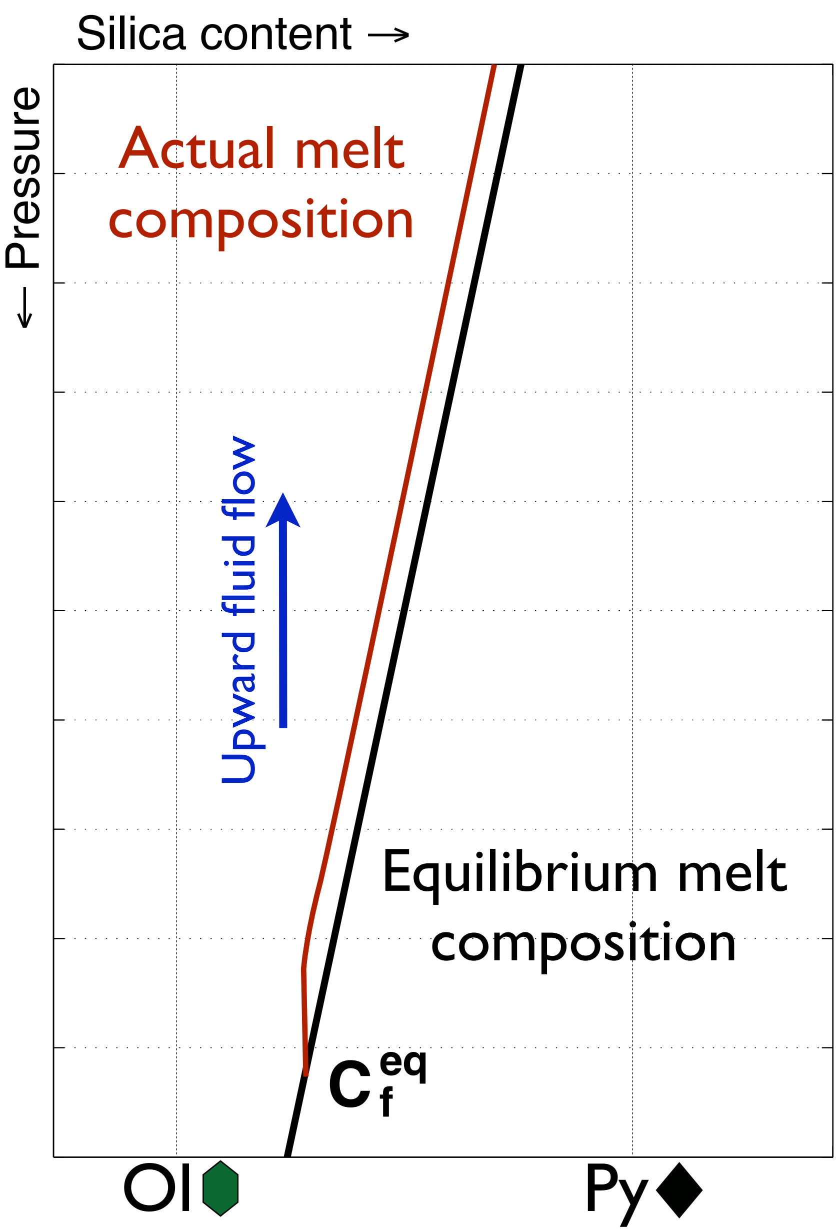
Reactive (flux) melting

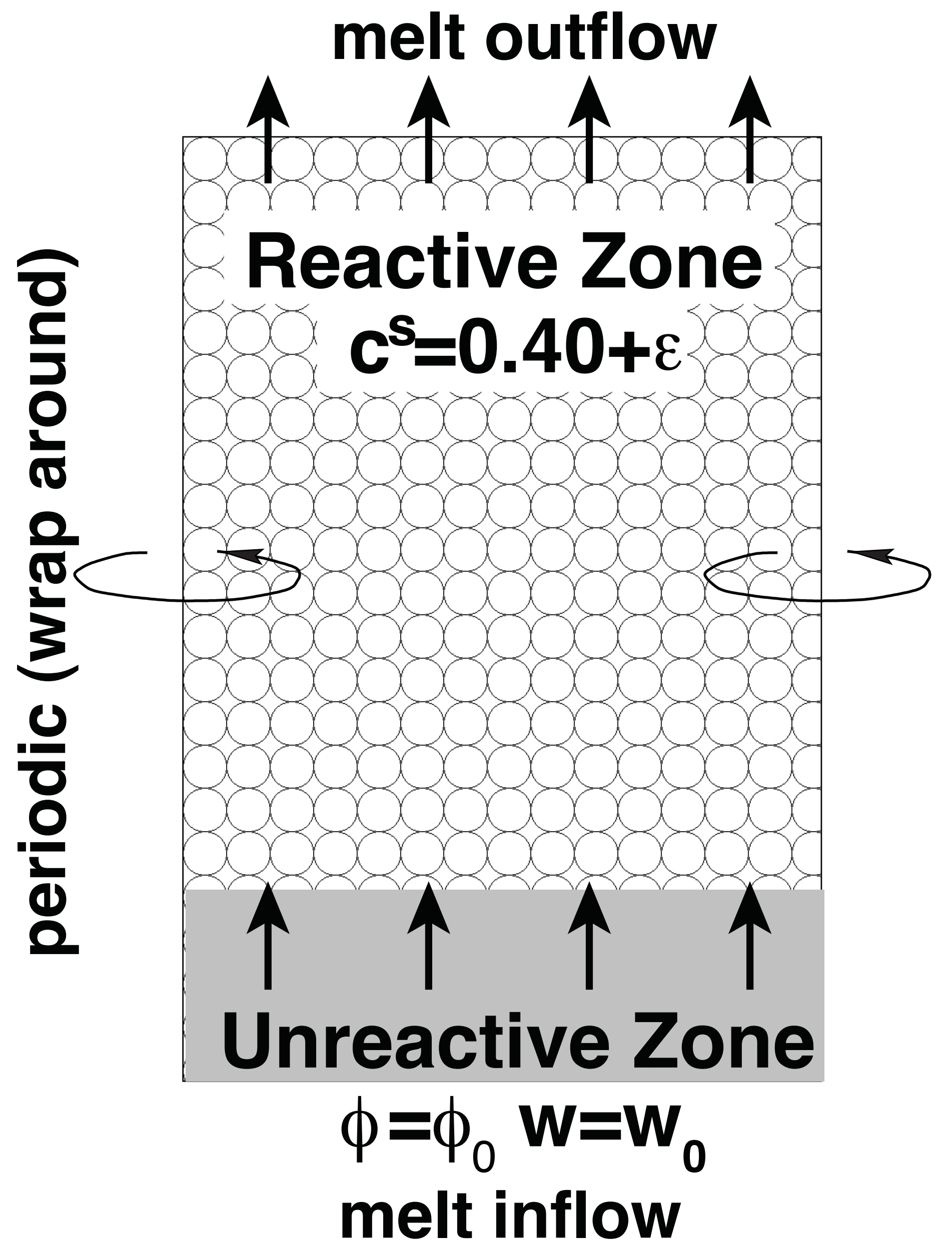
$$\Gamma \propto \phi w \frac{\partial C_f^{eq}}{\partial z}$$

# Problem 3: Think global, melts localize

Melting reaction:

Melt + Pyroxene + Spinel » Olivine + Melt



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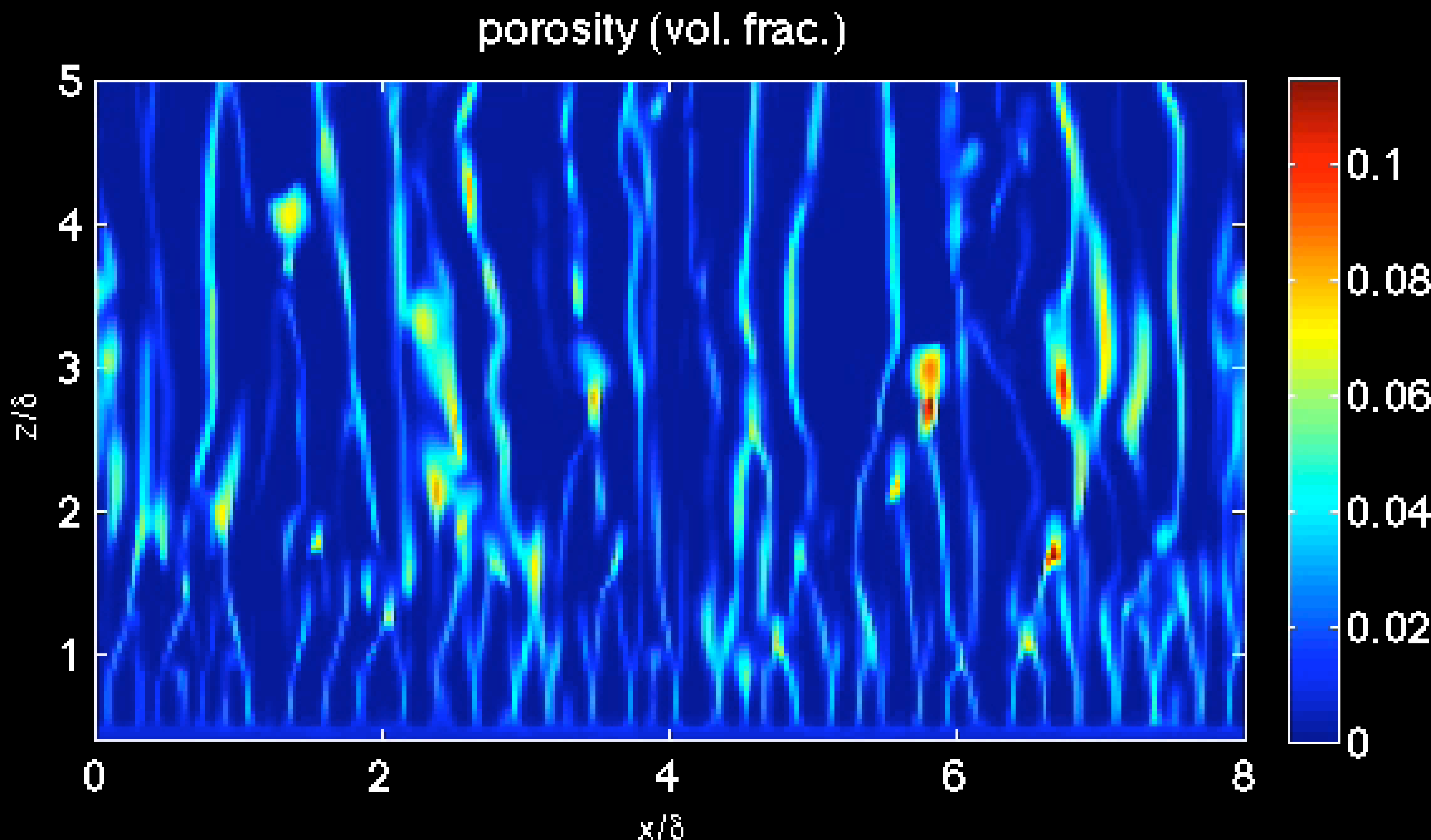

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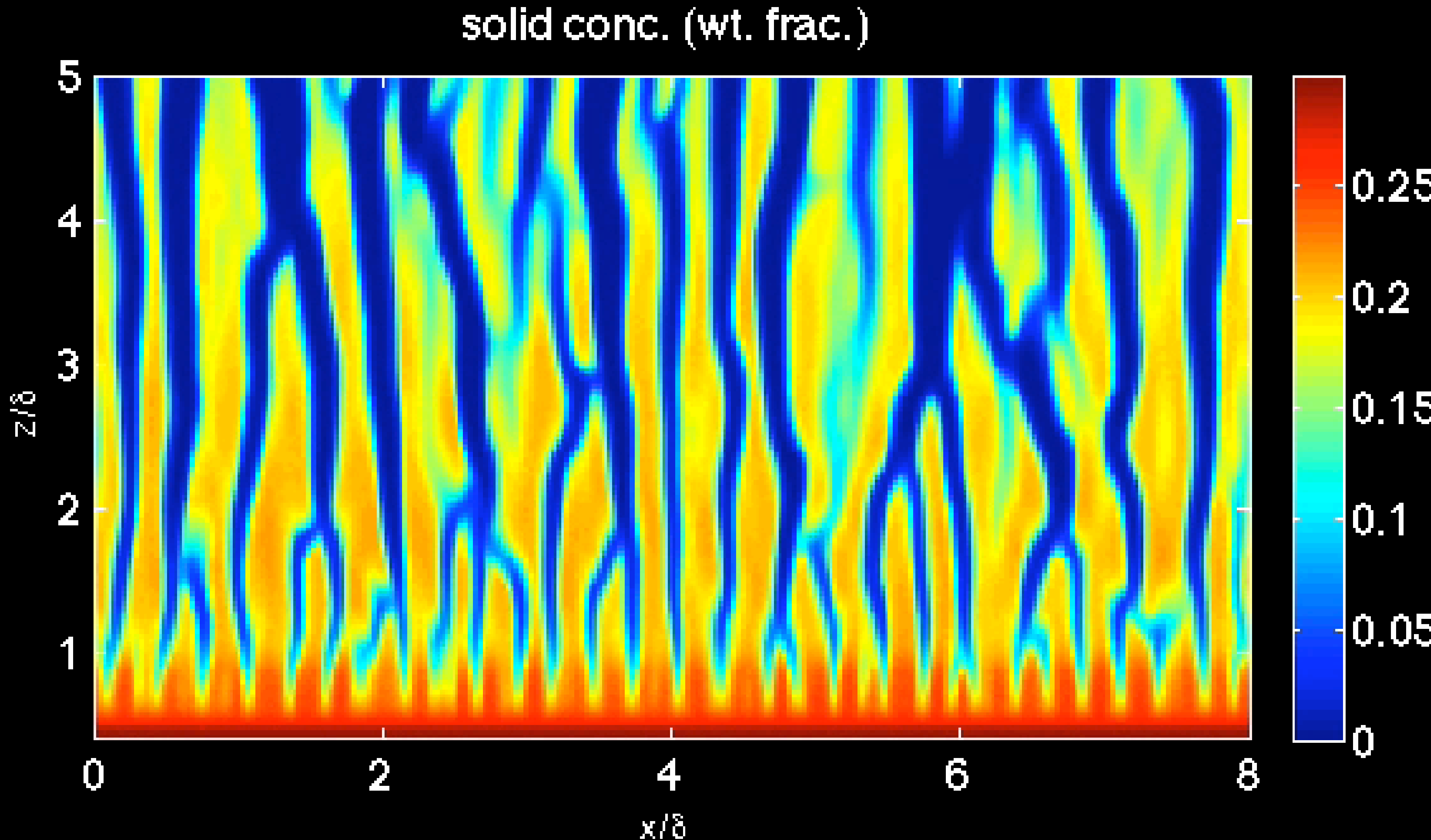
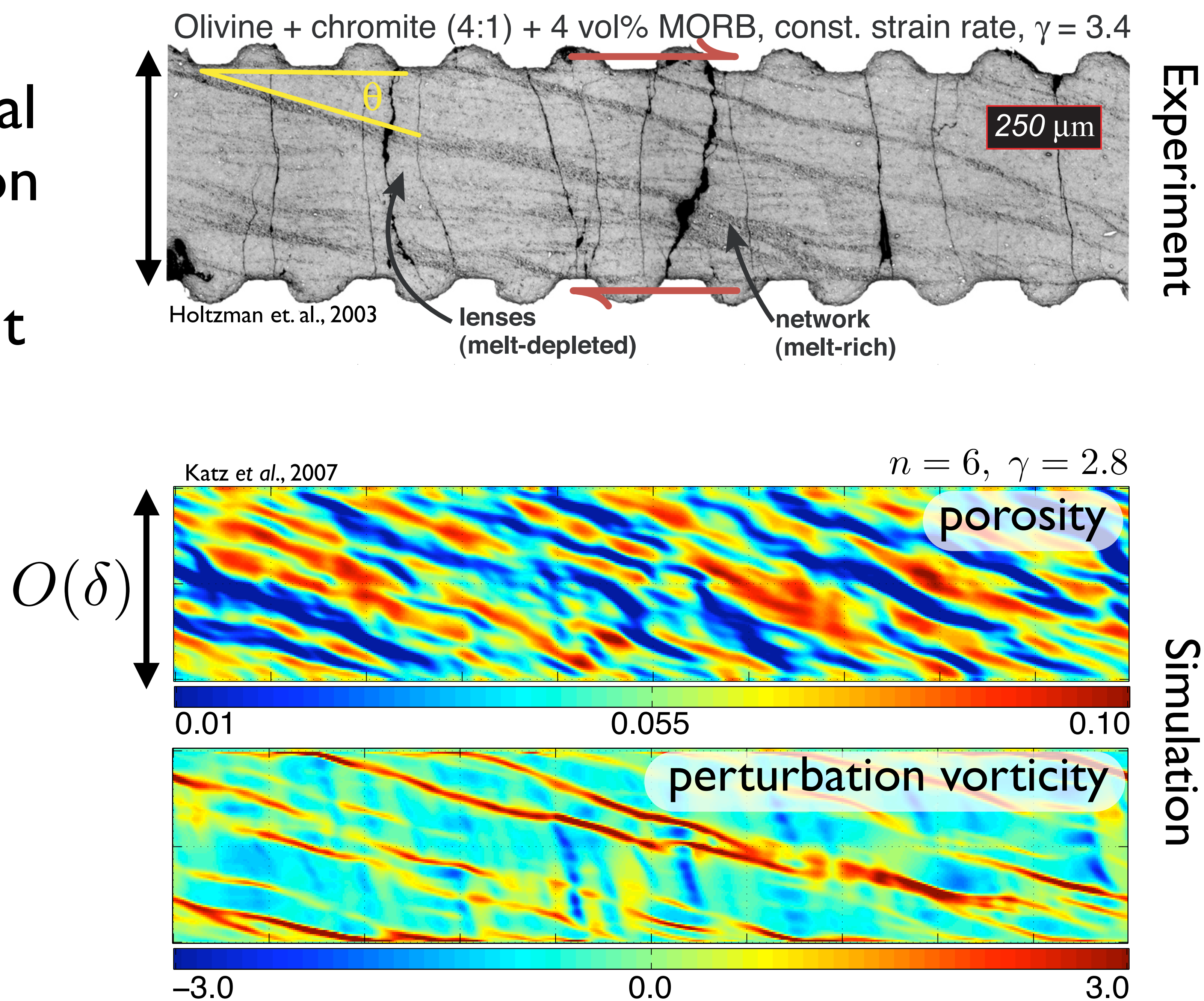


# Problem 3: Think global, melts localize

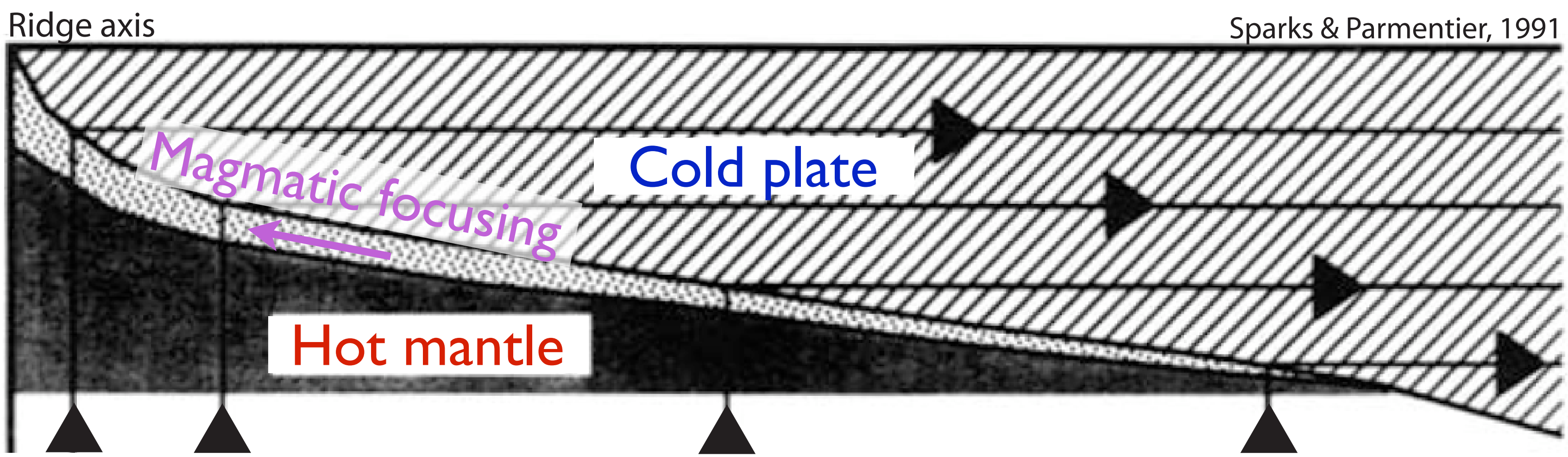
Mechanical localization may be important too.



# Problem 4: Mantle melts and freezes

Typically:

- Adopt or create a parameterization of melting experiments from the literature.
- Slap it into your code.
- Separate rules to handle freezing.

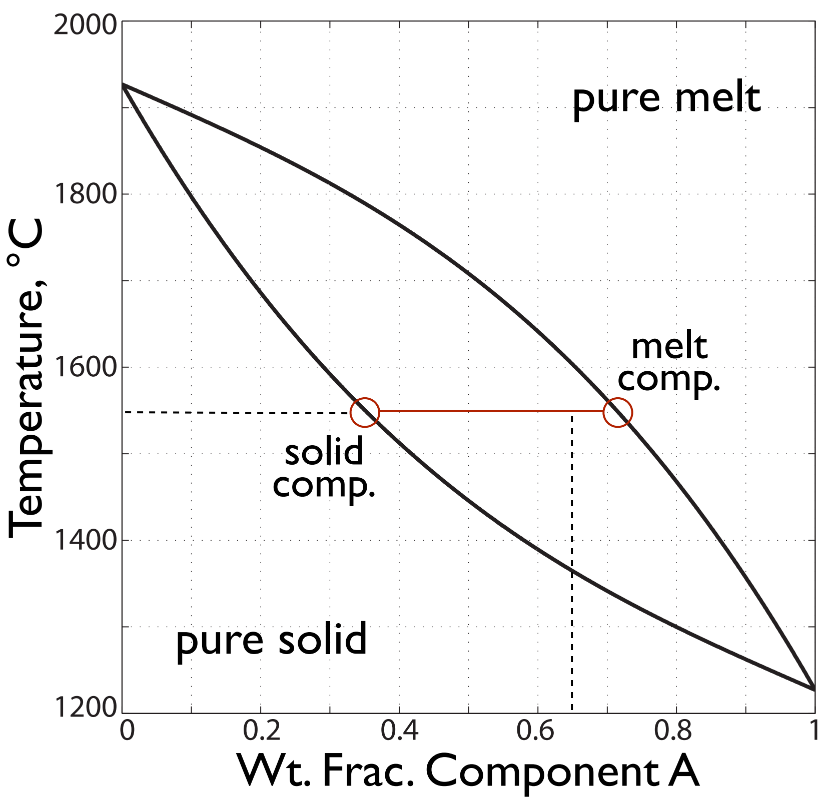



Better:

- Melting and freezing according to the same theory.
- Thermodynamically consistent.

# Problem 4: Mantle melts and freezes

*Enthalpy Method:*




- Assume thermodynamic equilibrium point-wise in the domain.
- Prescribe a phase diagram.
- Conserve enthalpy and bulk composition.
- Porosity, temperature & phase compositions given by constitutive equations.
- $\phi = f_1(H, C)$

  $\mathcal{T} = f_2(H, C)$

  $C_f = f_3(H, C)$

  $C_m = f_4(H, C)$

# Problem 4: Mantle melts and freezes

Use:

| | |
|---|---|
| $H = L\phi + c_P T$ | def. of bulk enthalpy |
| $dh = c_P dT + \rho^{-1}(1 - \alpha T)dP$ | total deriv. enthalpy |
| $C = \phi C_f + (1 - \phi)C_m$ | two components |

internal energy; advection of sensible heat; advection of latent heat; diffusion

$$\rho \frac{\partial H}{\partial t} + \rho c_P e^{\frac{\alpha g z}{c_P}} \boldsymbol{\nabla} \cdot \bar{\mathbf{v}} \mathcal{T} = \rho L \boldsymbol{\nabla} \cdot (1 - \phi) \mathbf{v}_m + k e^{\frac{\alpha g z}{c_P}} \nabla^2 \mathcal{T}$$

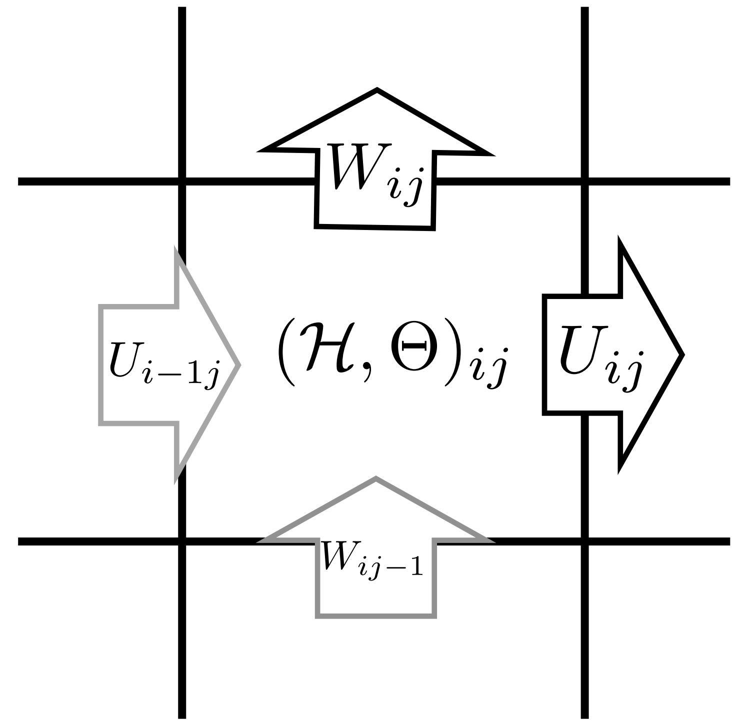
bulk composition; advection; diffusion

$$\frac{\partial C}{\partial t} + \boldsymbol{\nabla} \cdot \phi \mathbf{v}_f C_f + \boldsymbol{\nabla} \cdot (1 - \phi) \mathbf{v}_m C_m = \mathcal{D} \boldsymbol{\nabla} \cdot \phi \boldsymbol{\nabla} C_f$$

# Problem 4: Mantle melts and freezes

- Finite volume discretization on a staggered grid.
- Timestep (w/ Picard iteration):

```
Do n times {
  Solve cons. of energy & composition
  Solve cons. of momentum
}
```




Solve nonlinear systems on ~8 processors with Newton-Krylov-Schwartz (ILU preconditioner) provided by PETSc.
http://www.mcs.anl.gov/petsc

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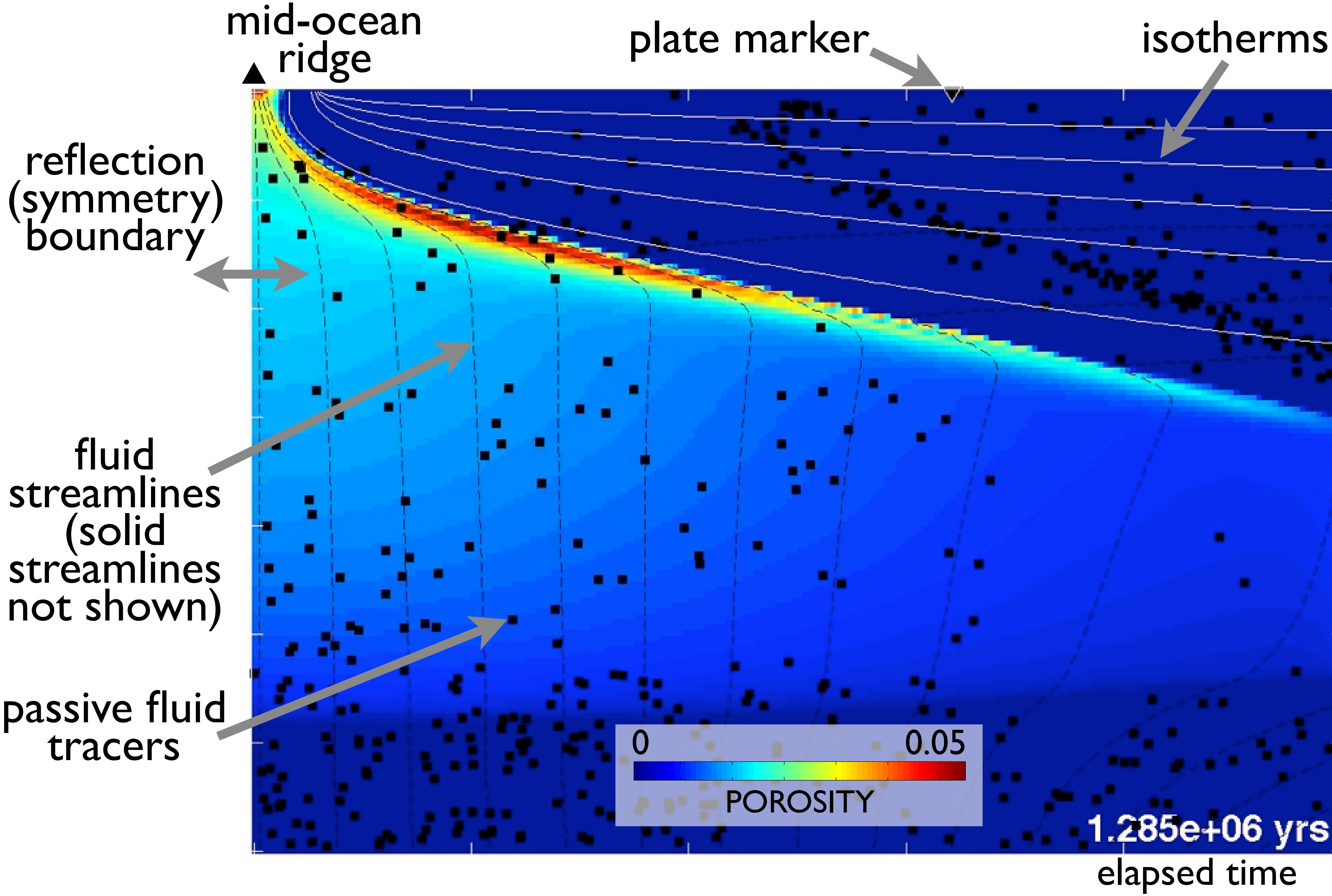

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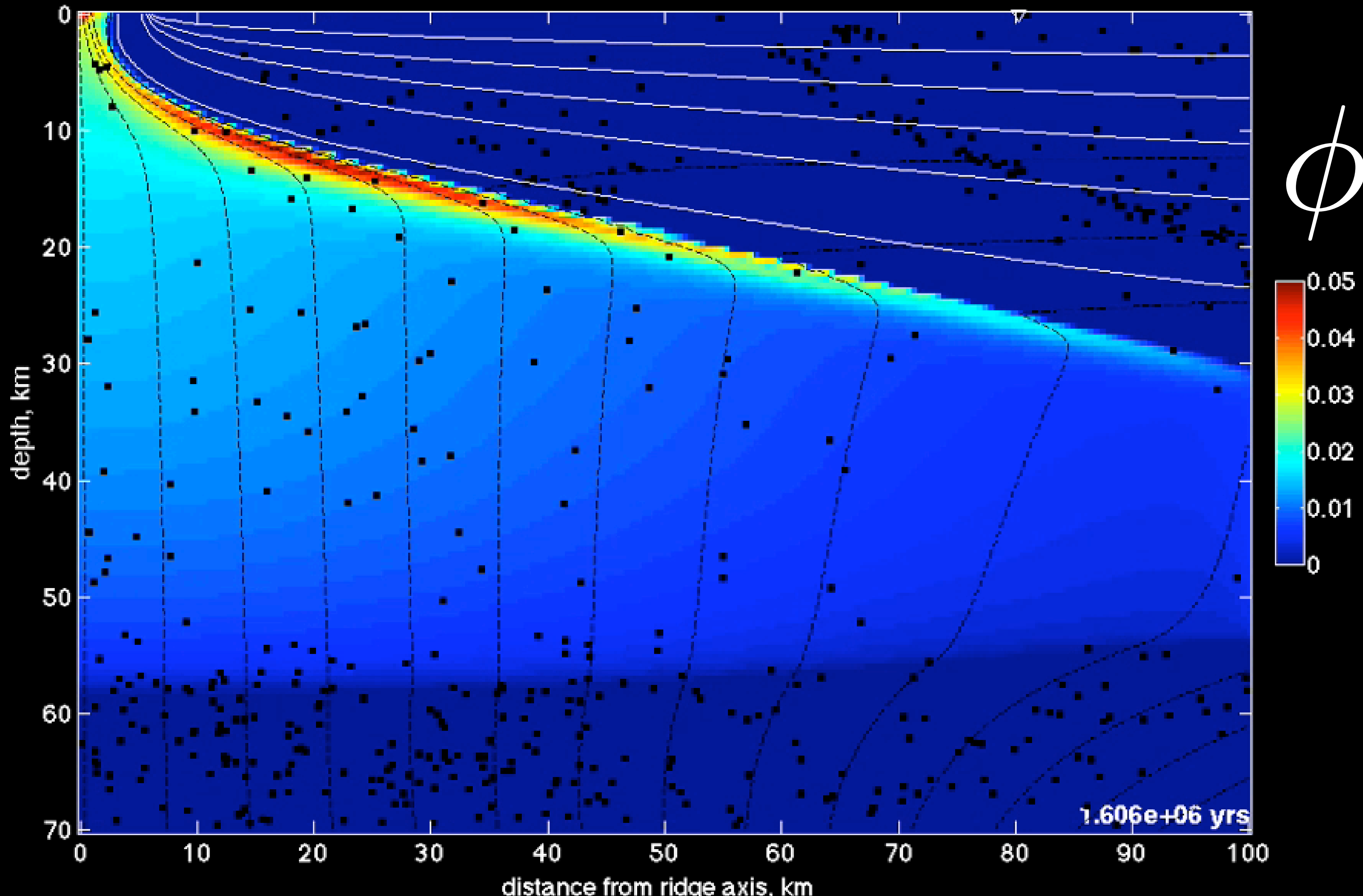

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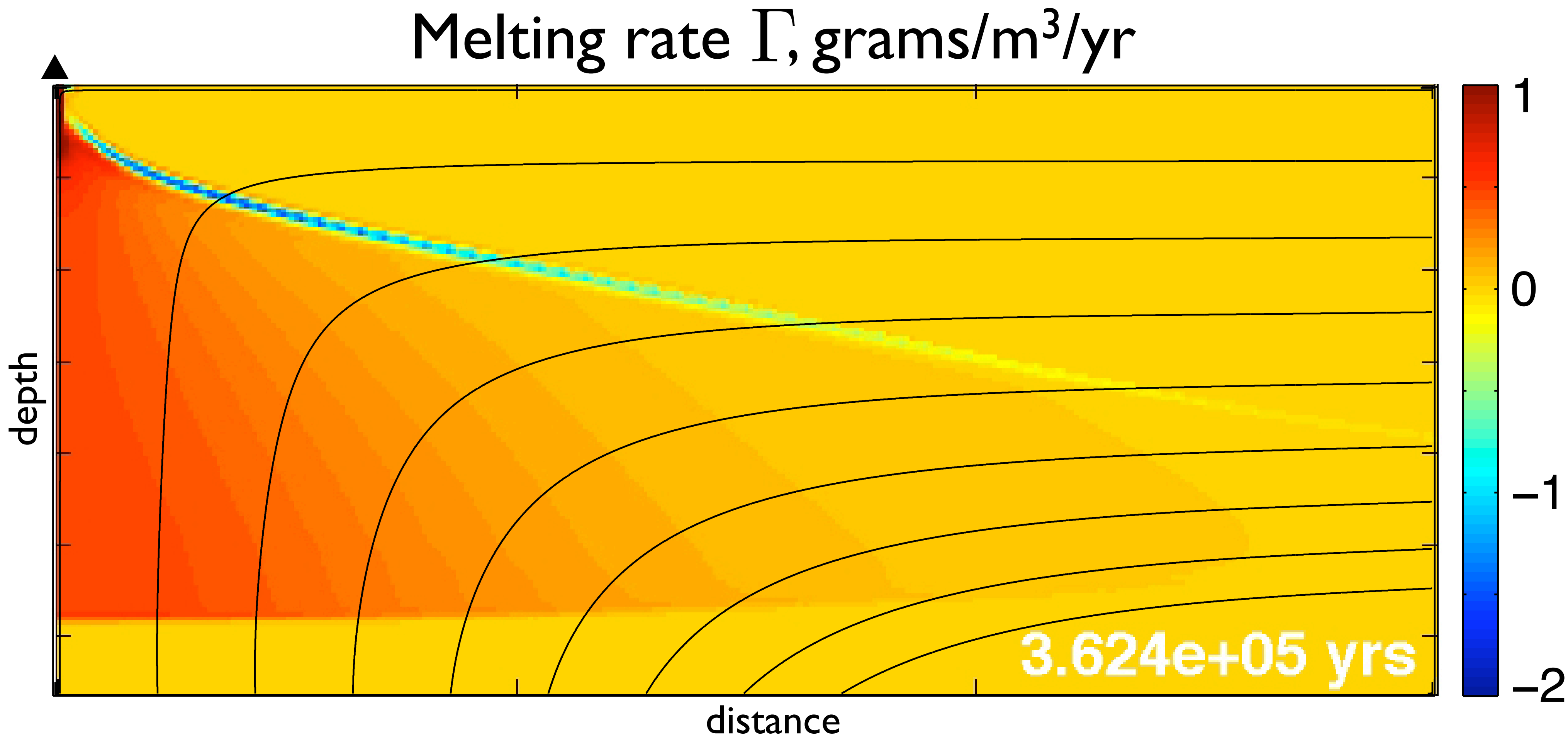


# Problem 5: Separation of time-scales

Mantle-- $W_0 \approx 10^{-2} - 10^{-1}$ m/yr

Magma-- $w_0 \approx 10^{0} - 10^{1}$ m/yr

$$w_0/W_0 \approx 100$$

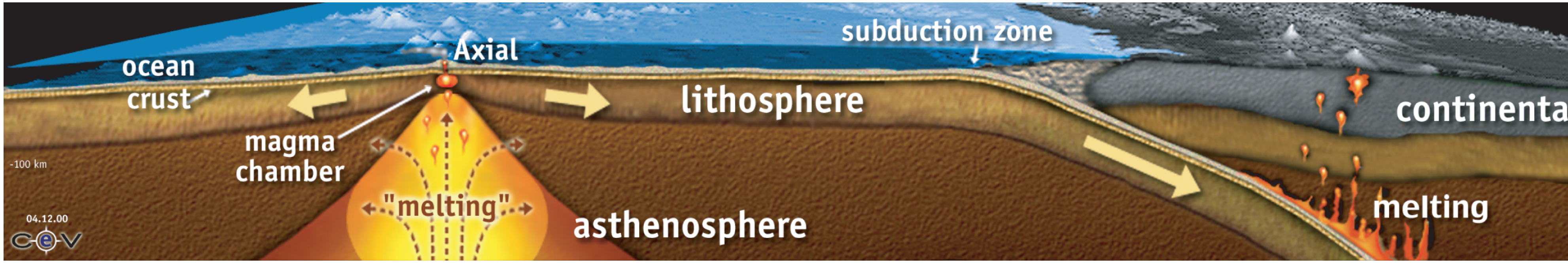
A typical time step on 8 processors of Jazz (Argonne) with ~20k DOF/node:

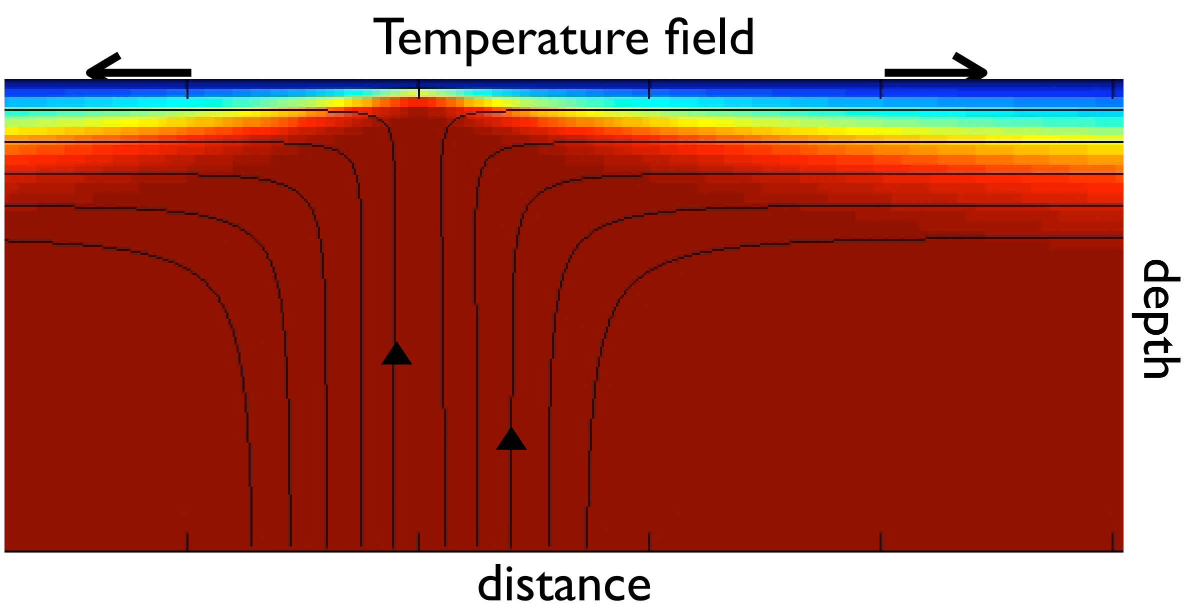
```
----------------------------------------------------------------
  0 SNES Function norm 2.892345380904e-03
  1 SNES Function norm 6.134063703318e-07
 HC solve: CONVERGED_FNORM_ABS, KSP/Nstep: 1.0
  0 SNES Function norm 7.777350795075e-03
  1 SNES Function norm 1.282132776270e-07
 PV solve: CONVERGED_FNORM_ABS, KSP/Nstep: 520.0
Step: 42 (33.8 sec), time: 4.73e+04, dt: 2.42e+02 dtmax: 2.42e+02 yrs
----------------------------------------------------------------
```

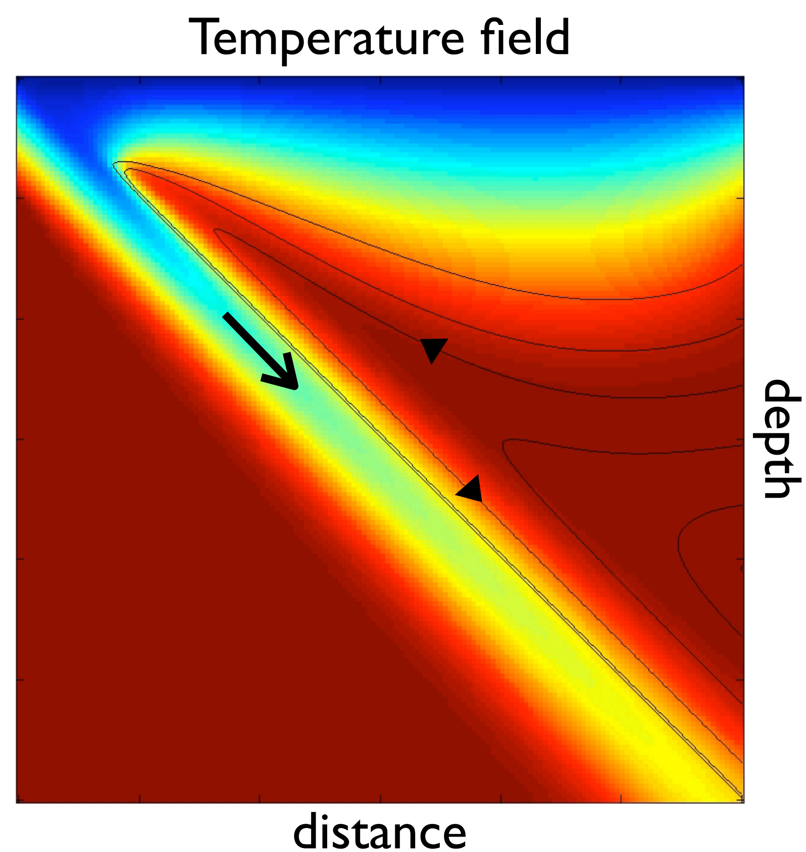
# Problem 5: Separation of time-scales

- Solution: update flow field pressure after every N updates of enthalpy and bulk composition?
- Does not converge.
- Solution: split compaction equation from momentum and continuity??
- Solution: use field-split preconditioner for flow/pressure solve??
- Solution: Find a better preconditioner/solver???
- I should try these.

# Problem 6: Subduction isn't square



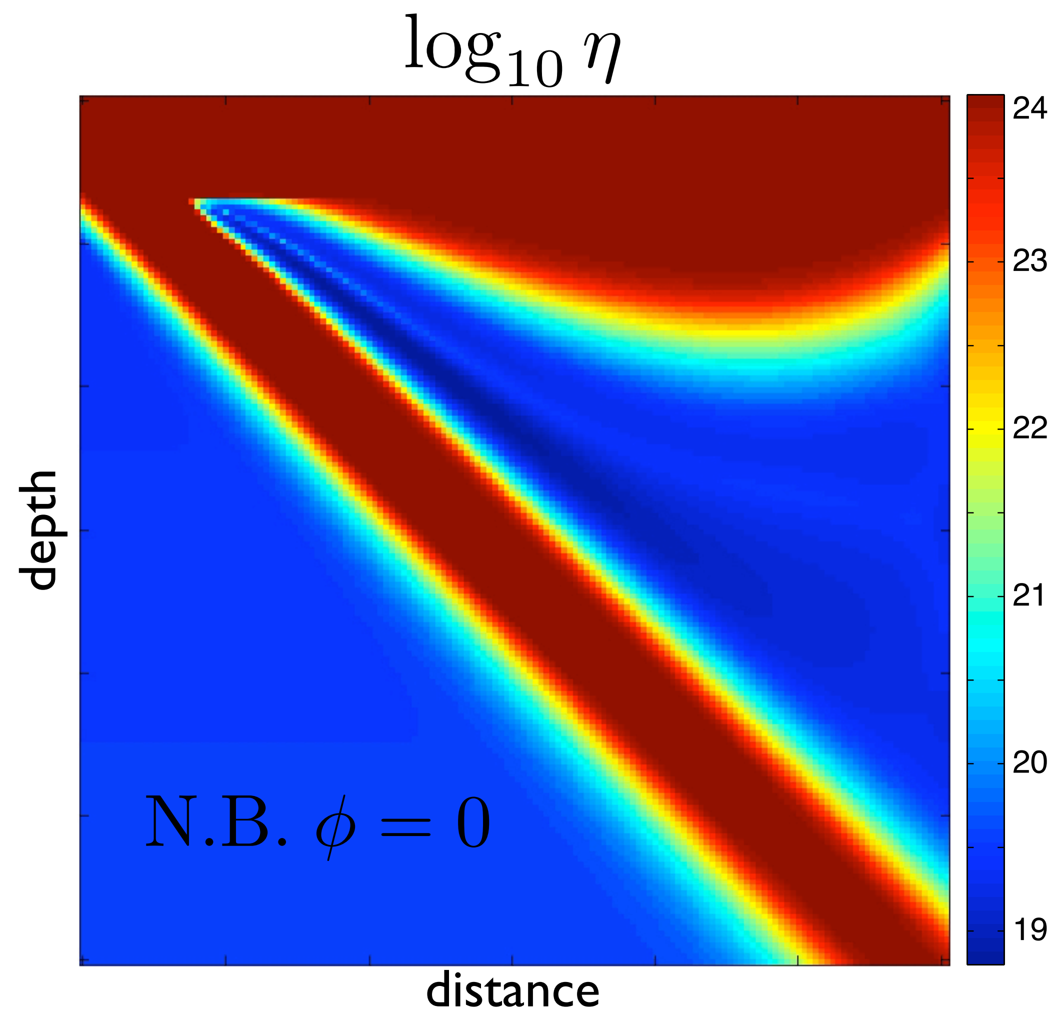



# Problem 7: Viscosity will vary (a lot)

Matrix viscosity depends on:

- Temperature
- Strain rate
- Porosity
- (Pressure)
- (Water content)
- etc...




$$\eta = \eta_0 \mathrm{e}^{\left(\frac{E^*}{RT} - \alpha\phi\right)} \dot{\epsilon}_{II}^{\frac{1-n}{n}}$$

# Many more problems...

- Non-constant viscosity in tectonic-scale simulations.
- Multi-scale simulations: magma embedded in mantle convection.
- Thermodynamically consistent disequilibrium melting.
- Optimal solvers (i.e. multigrid).
- etc.

# Conclusions

- Magma dynamics is a physically rich and computationally challenging field.
- Currently (most) important research area for understanding internal dynamics of Earth.
- Plenty of fun, tractable problems to tackle.
- Progress is being made. Stay tuned or get involved.