Extended Finite Element Methods for Crustal Deformation Simulation

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Earthquake rupture

Verification

Fault System Simulation

Conclusions and Discussion

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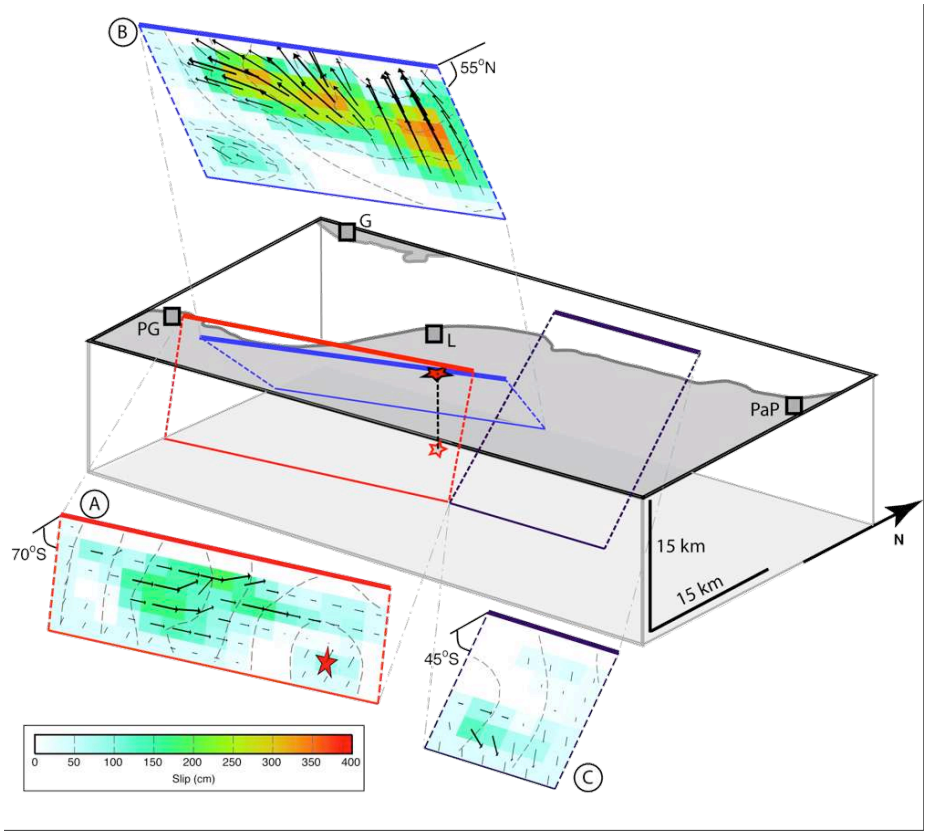
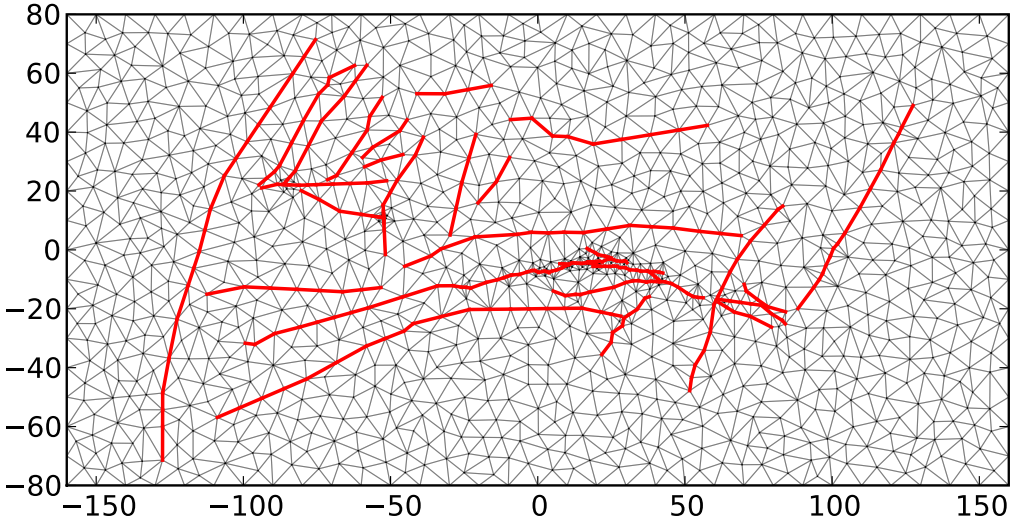


Haiti Earthquake
Hayes et al, Wednesday morning

The biggest issue inhibiting the inclusion of fault system geometry in crustal deformation and earthquake simulations is fault meshing.

- Generating meshes which both conform to fault geometry and have good properties for simulation is hard.
- Current process is iterated by hand.
- Fault geometry is uncertain – are solutions robust to fault geometry variation?
- Fault geometry is uncertain – what is the definition of fault?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Accuracy in FEM simulations

In order for FEM solutions to converge, we need the following (approximability):

$$\inf_{v \in \mathcal{V}^h} ||u - v|| \to 0 \text{ as } h \to 0$$

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Take advantage of *a priori* knowledge of fault geometry and the resulting solutions to construct basis functions which better represent our expected solution u.

Partition of Unity FEM (Melenk and Babuska '96)

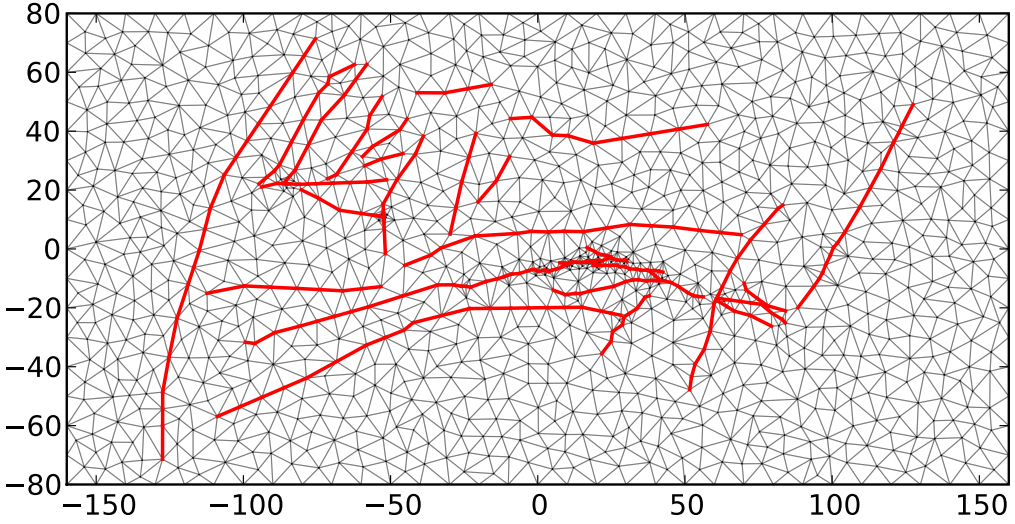
Given:

- Partition of unity function space: $\{\phi_i\} \quad s.t. \quad \sum_i \phi_i = 1$ on entire domain
- Local neighborhoods $\Omega_i = support(\phi_i)$

Introduce:

- Local extension functions F_j which incorporate properties of the physics
- Local function spaces $\mathcal{V}^i = \{1\phi_i\} \bigcup \{F_j\phi_i\}_j$
- Approximation space: $\mathcal{V} = \bigcup_j \mathcal{V}^i$

Can show that, if the $\{F_j\}_j$ functions well-approximate the solution on Ω^i, then the solution for $\inf_{v \in \mathcal{V}} ||v - u||$ converges as quickly as approximating a smooth solution with the $\{\phi_i\}$.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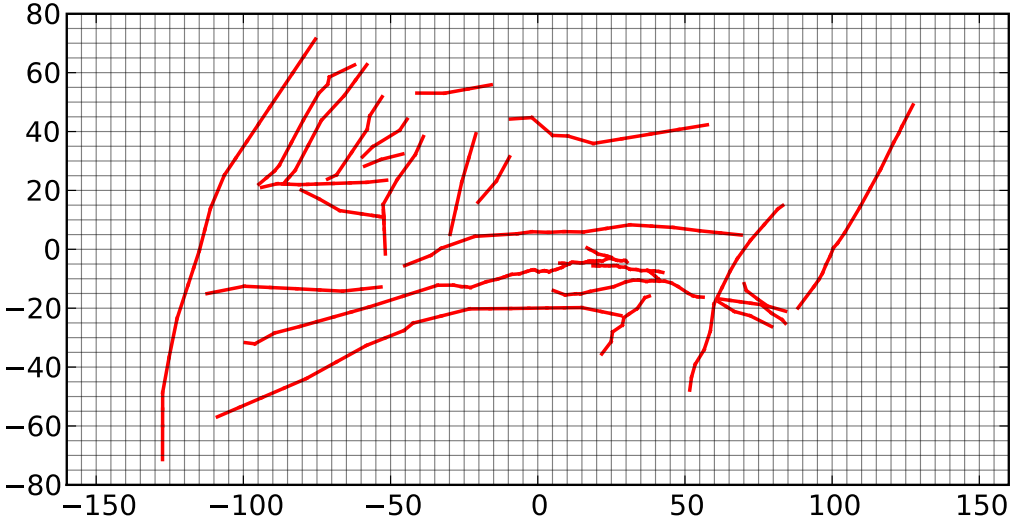
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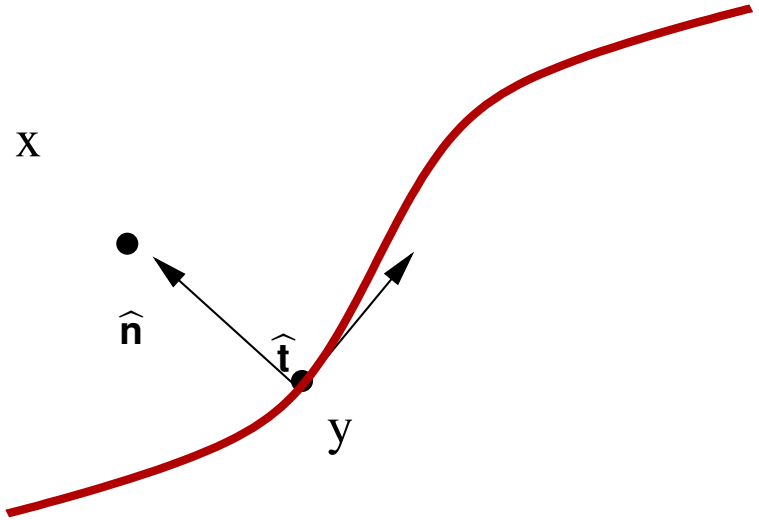
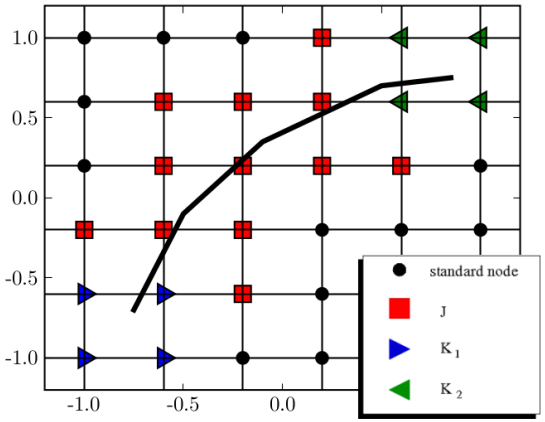
Heaviside extensions

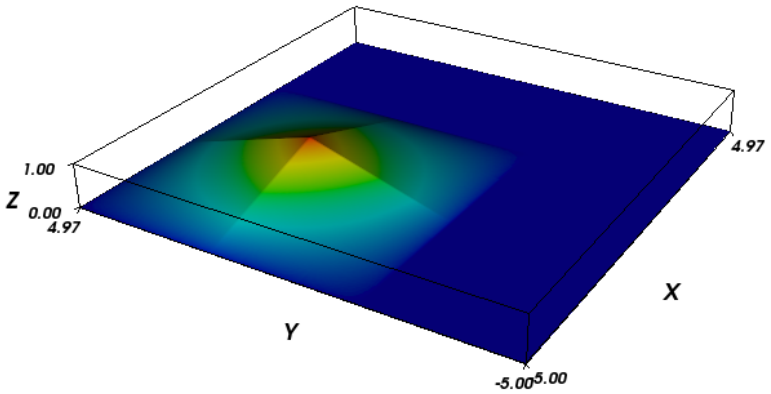
For each fault Γ^k, introduce a Haar function extension.

$$F^k(x) \equiv \begin{cases} 1 & \text{if } (x-y)\cdot\widehat{\mathbf{n}}(y) > 0 \\ -1 & \text{otherwise} \end{cases}$$

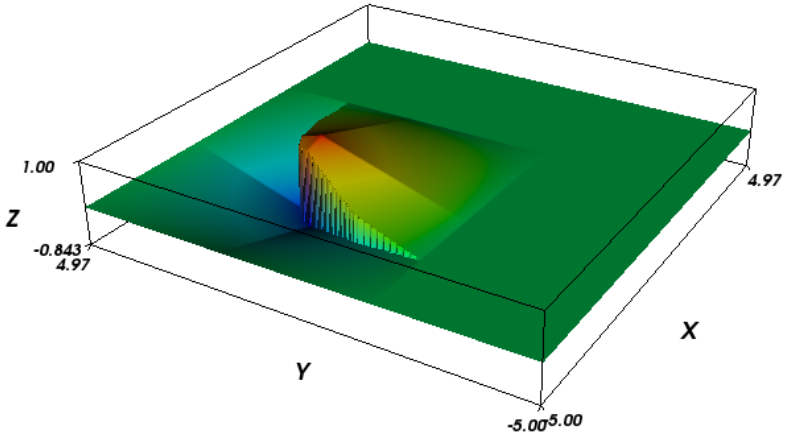
where $y \equiv \arg\min_{z\in\Gamma} |x-z|$







Standard Basis Function



Extended Basis Function

(Belytschko & Black, 1999)
(Dolbow, Moes, Belytschko 2000)

Tip extensions

At the tip of crack k, x^k_{tip}, introduce tip functions which match the stress singularity of elasticity (or whatever tip function you like).

$$F^k_{1-4}(x) \equiv \{\sqrt{r}\sin(\theta/2)$$
$$\sqrt{r}\cos(\theta/2)$$
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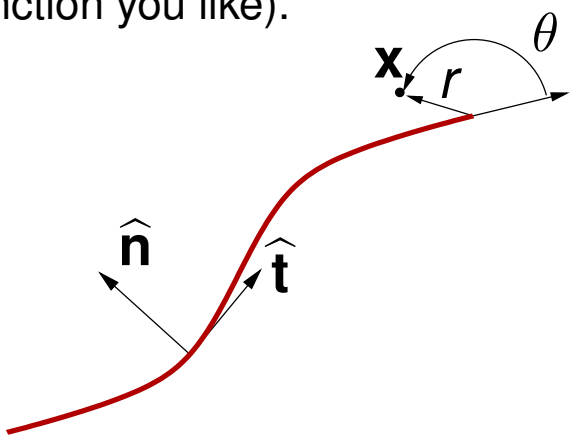
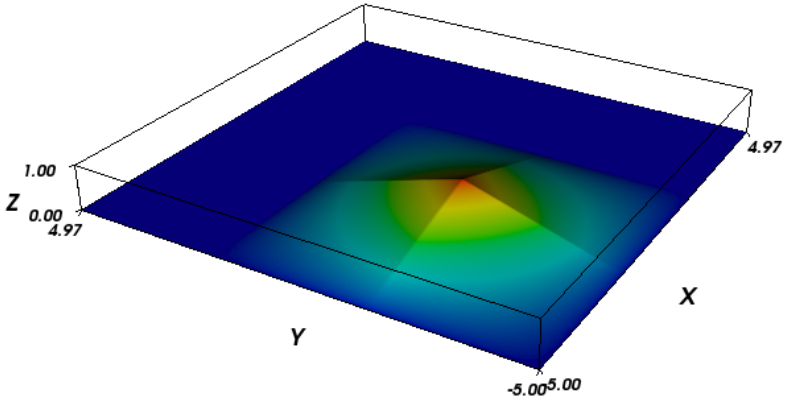


$r \equiv \left| x - x^j_{tip} \right|$

θ is the angle from the tangent of Γ^k at x^k_{tip} to $x - x^k_{tip}$



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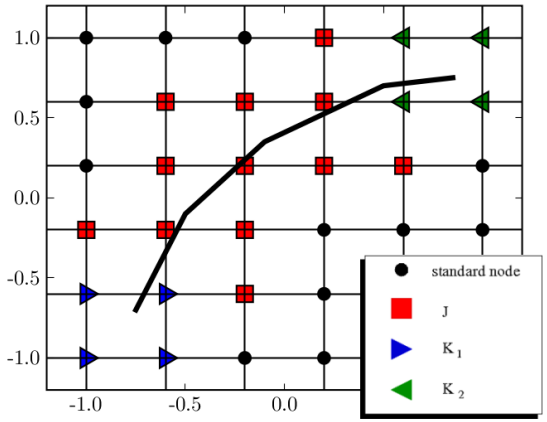

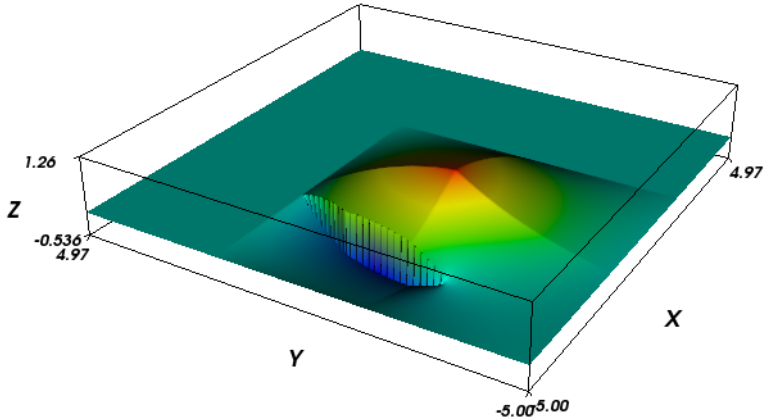

Tip-extended Basis Function

(Belytschko & Black, 1999)
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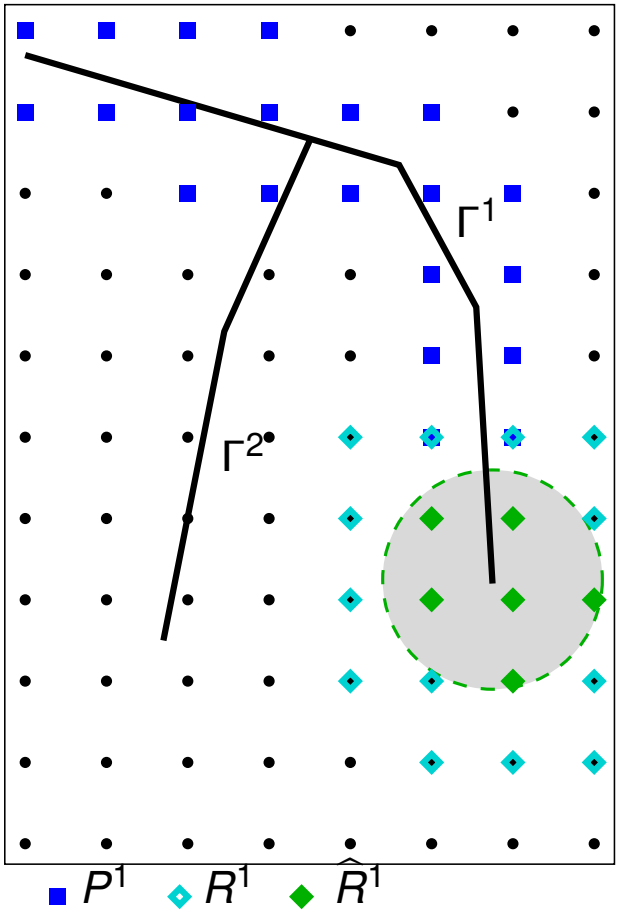
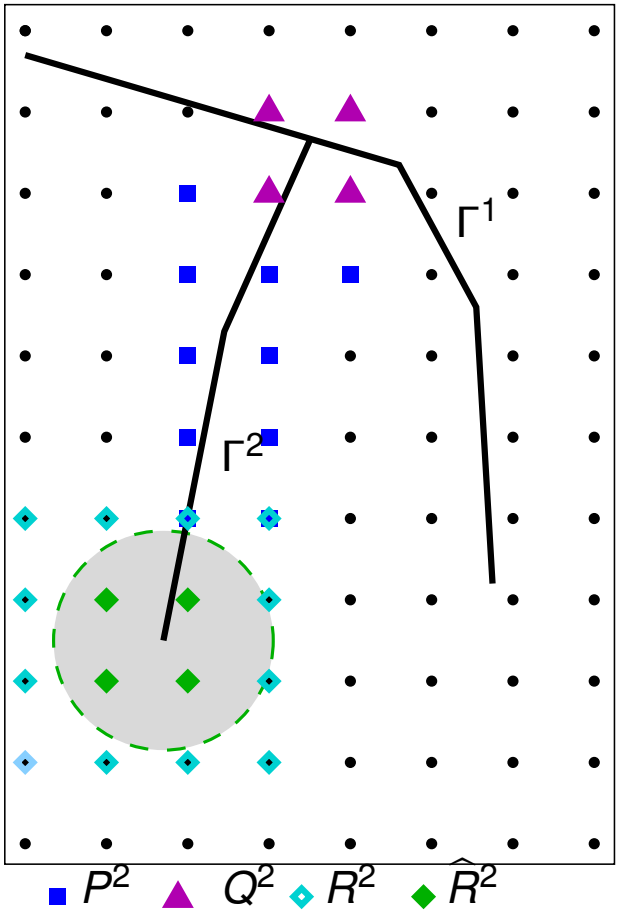


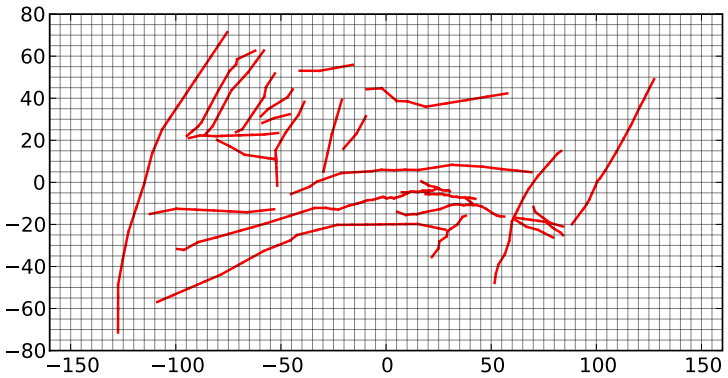
Extensions on crack 1



Extensions on crack 2

Implementation

- hybrid code – Python/Cython/C
- serial code, hard on memory
- leverages PETSc for solvers (all the work)
- leverages petsc4py to use PETSc
- prototype for inclusion in larger, community codes such as PyLith



Earthquakes

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma}(\mathbf{u})$$

- no interpenetration:

$$[\![\mathbf{u} \cdot \widehat{\mathbf{n}}]\!]_\Gamma = 0$$

- stick (Γ^D):

$$x \in \Gamma^N \; s.t. \; \left[\!\!\left[\frac{\partial \mathbf{u}}{\partial t}\right]\!\!\right]_{\Gamma^N} = 0 \Longrightarrow x \in \Gamma^D$$

$$\Rightarrow \left[\!\!\left[\frac{\partial \mathbf{u}}{\partial t}\right]\!\!\right]_{\Gamma^D} = 0$$

- slip (Γ^N):

$$x \in \Gamma^D \; s.t. \; \left|\widehat{\mathbf{t}}^T \cdot \boldsymbol{\sigma} \cdot \widehat{\mathbf{n}}|_{\Gamma^N}\right| > \left(\widehat{\mathbf{n}}^T \cdot \boldsymbol{\sigma} \cdot \widehat{\mathbf{n}}|_{\Gamma^D}\right) \mu_s$$

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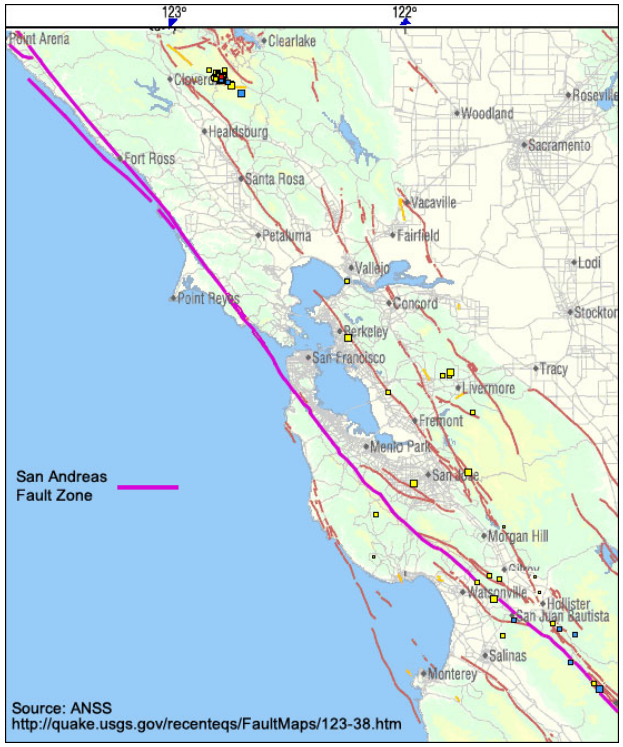

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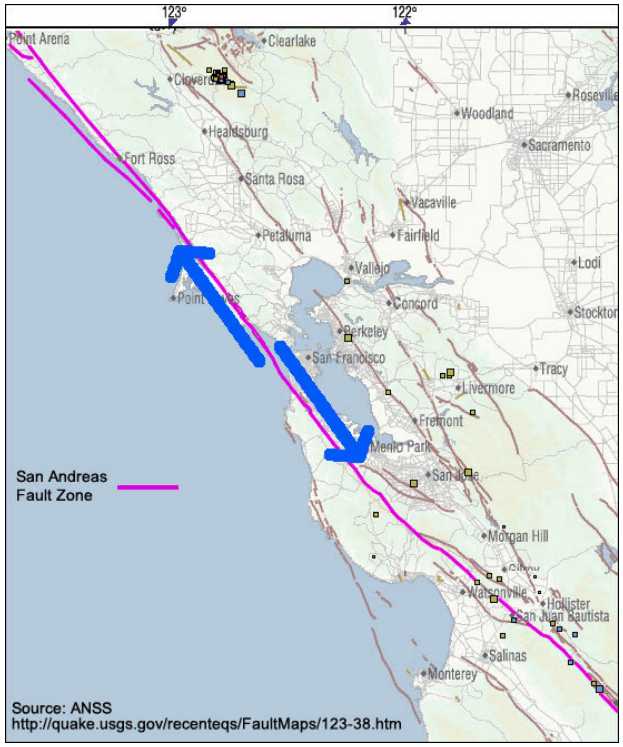

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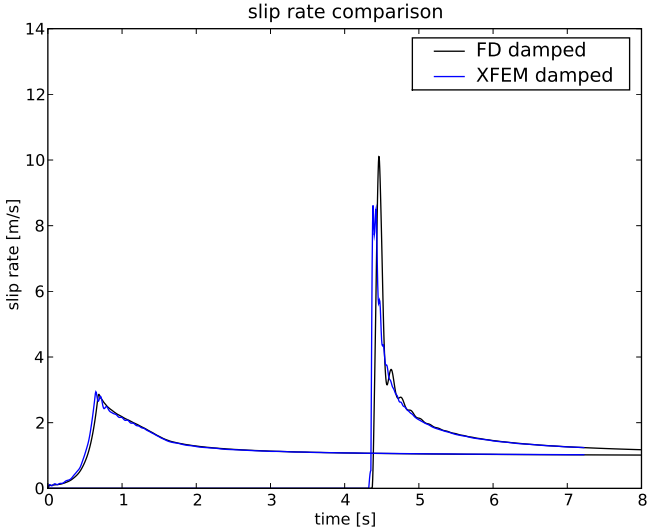
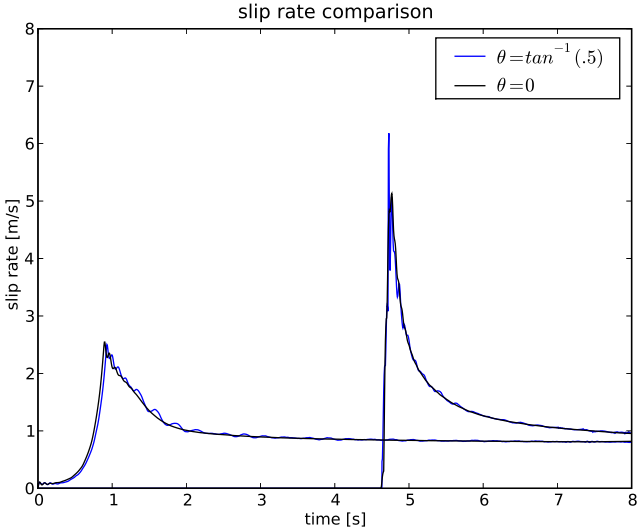
Verification: Mode III





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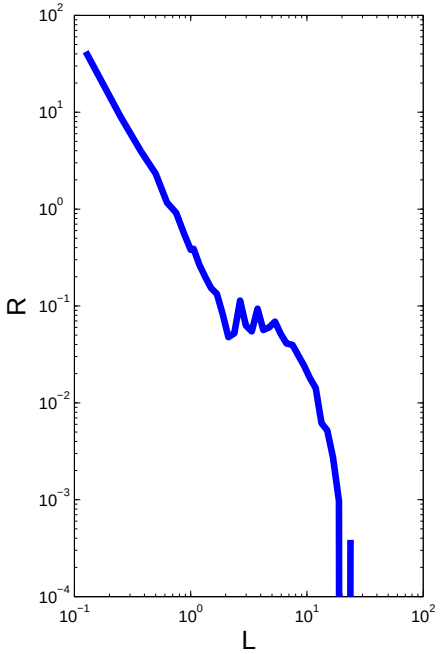
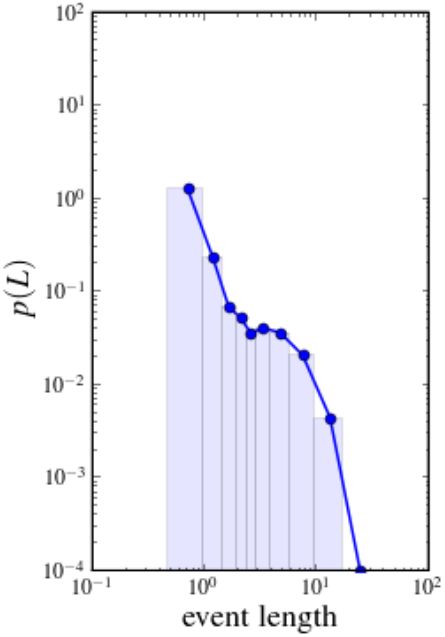
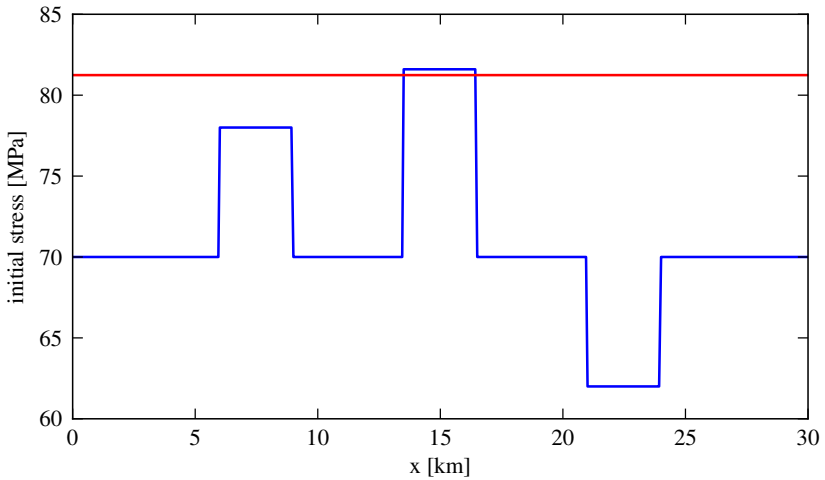
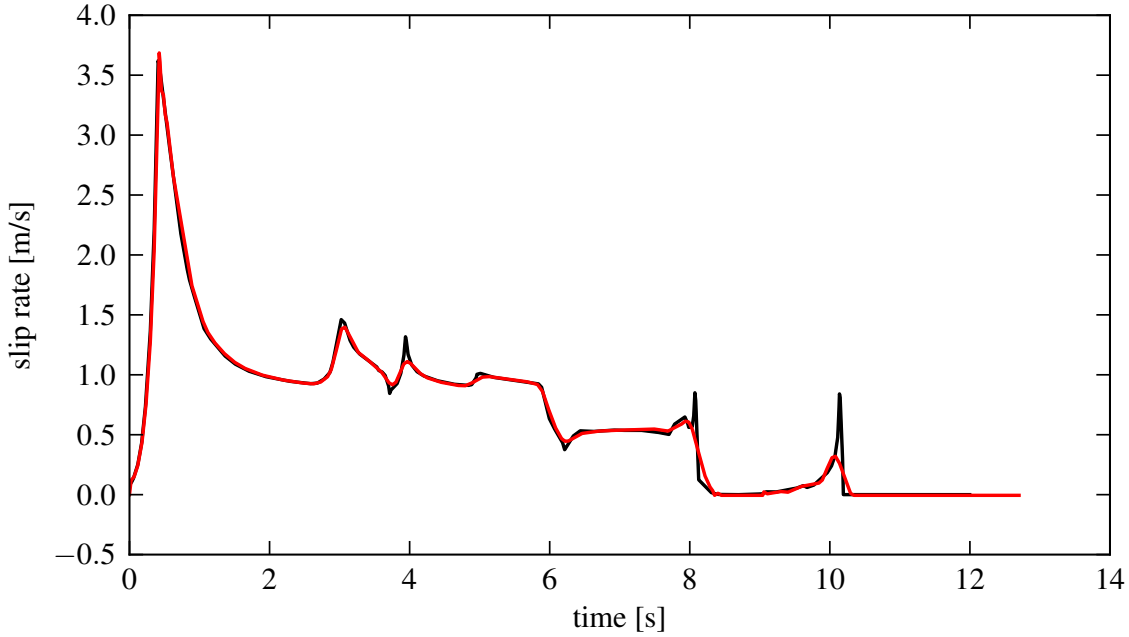




XFEM

Verification: Mode II, Source Dynamics Benchmark, TPV-205





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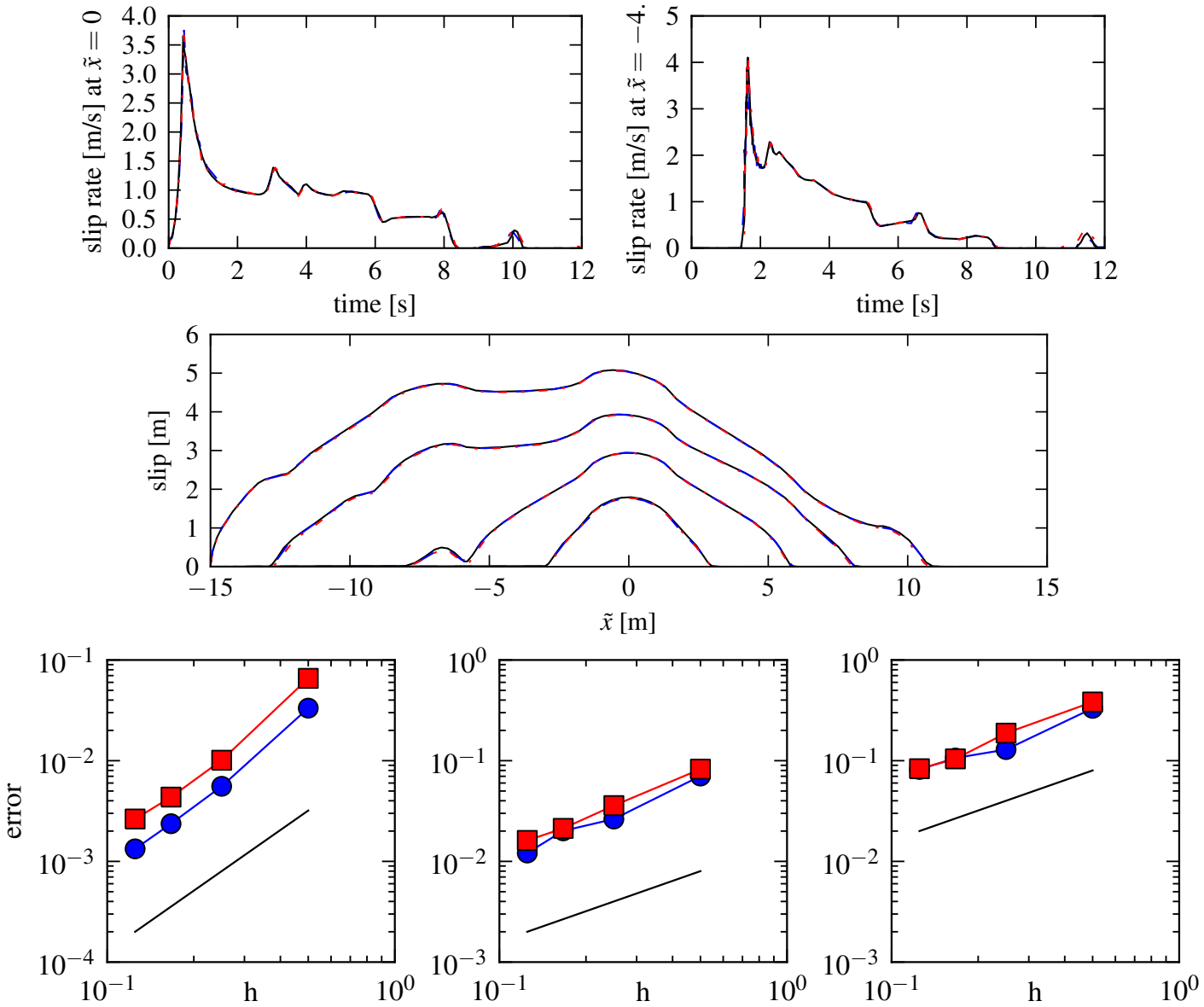
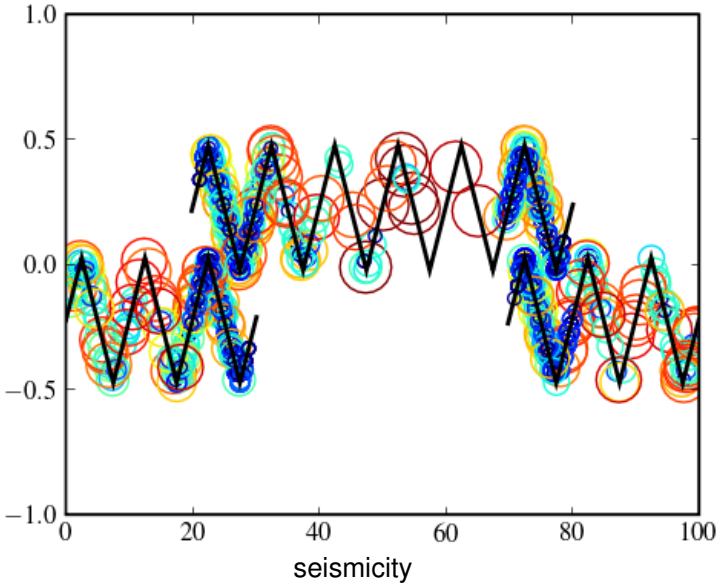
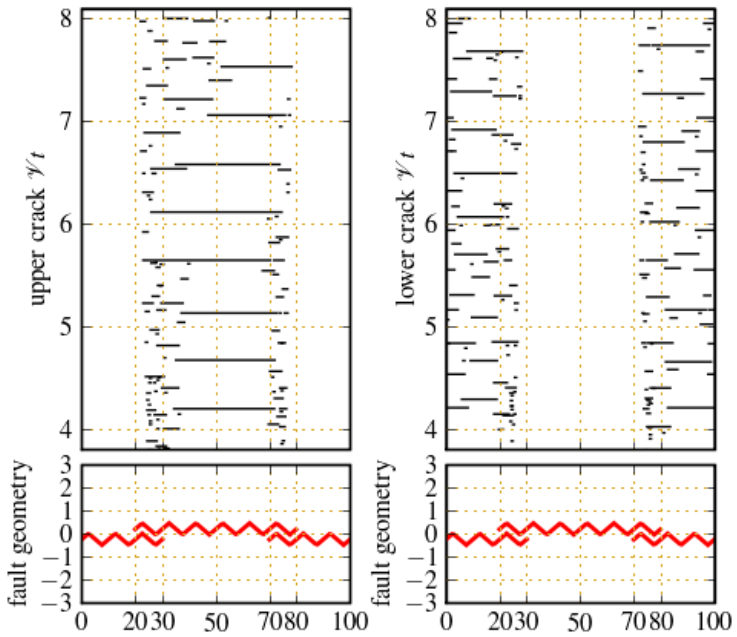
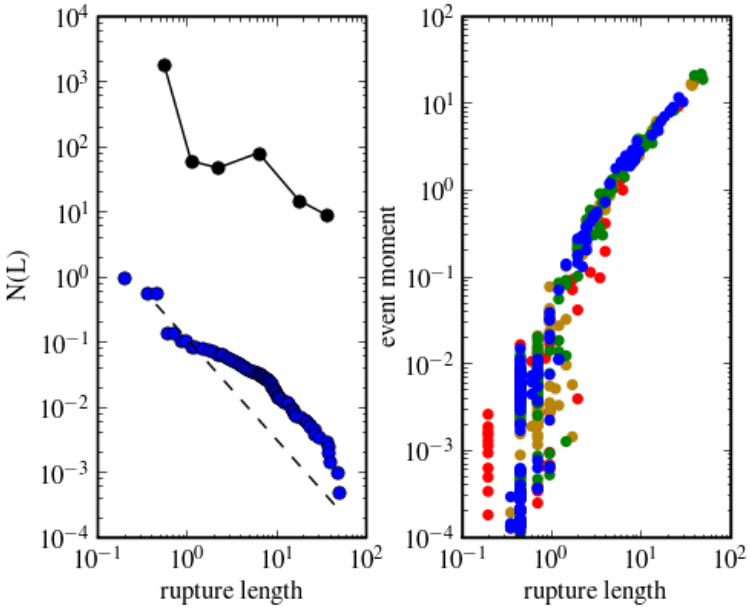
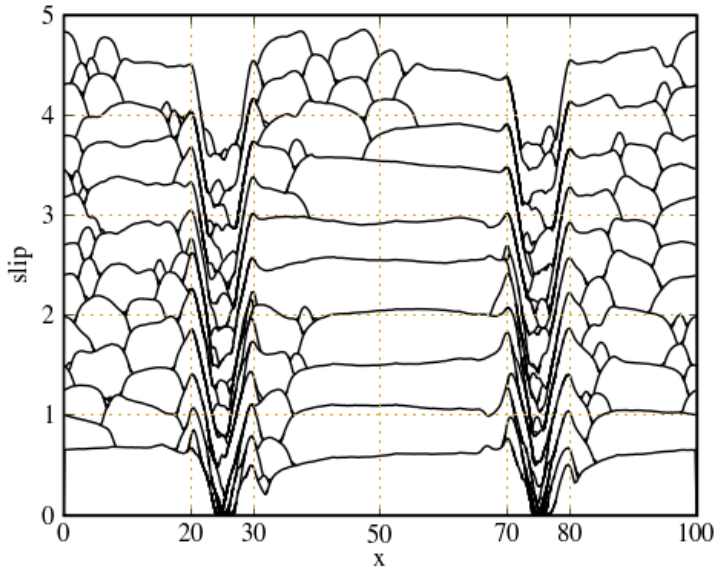


Mode III, Repeated Rupture

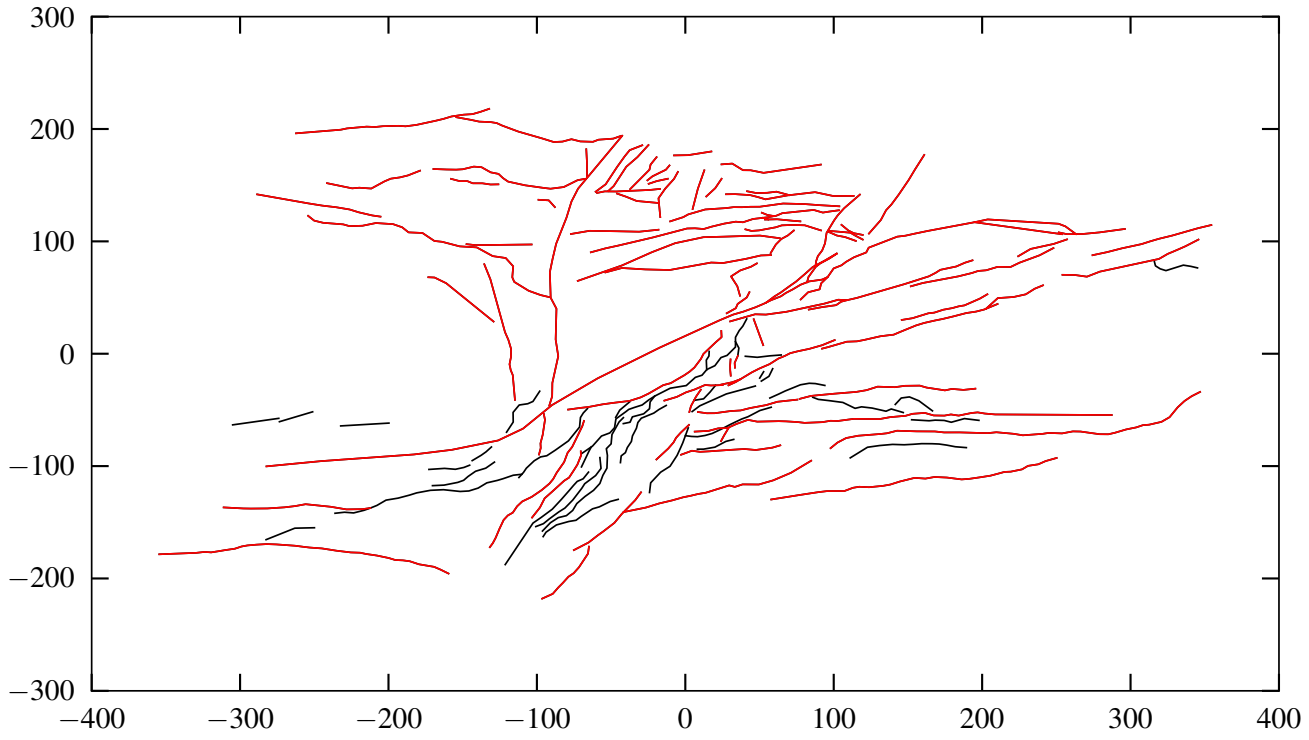








Mode II, CFM Surface Traces



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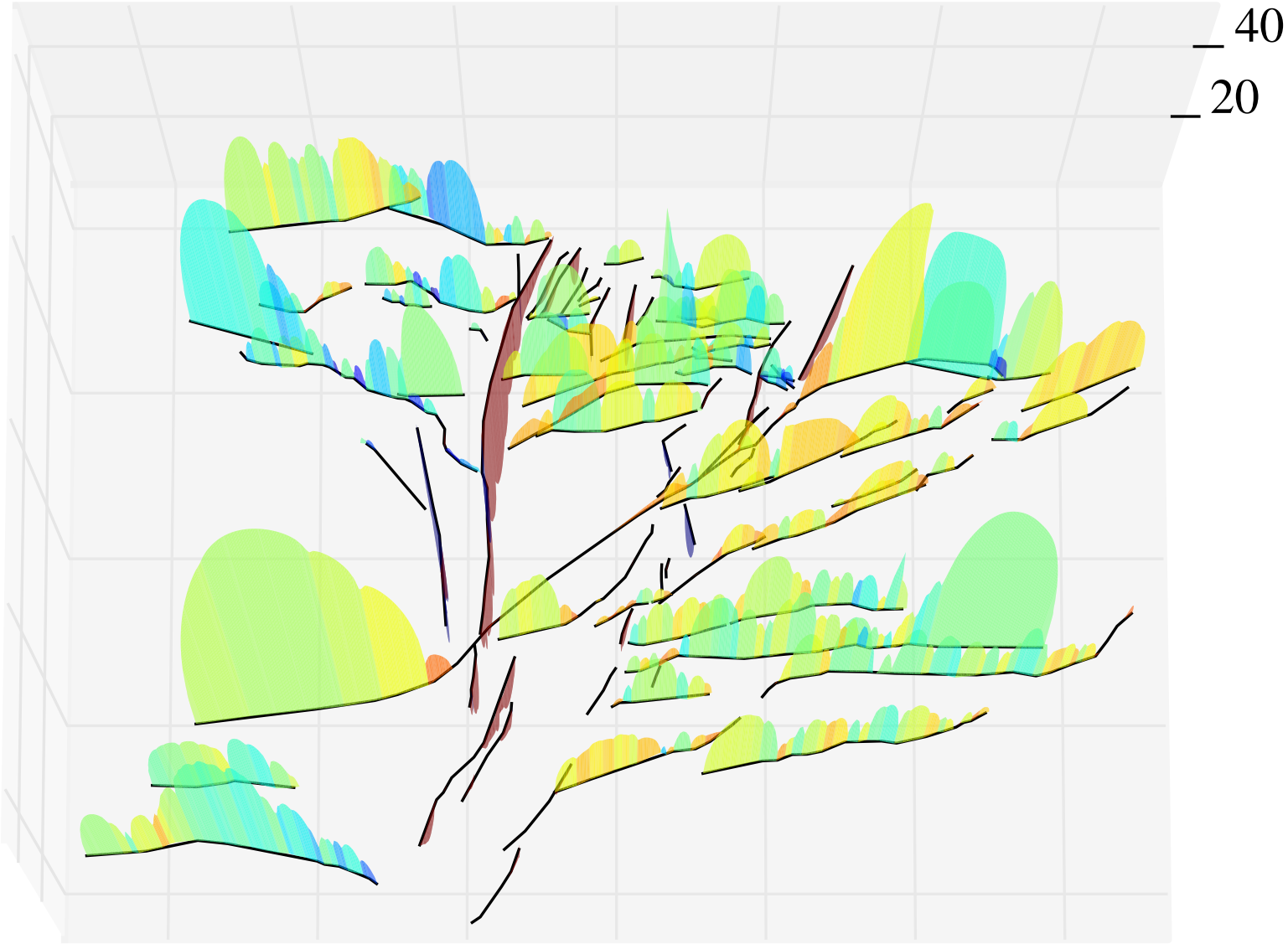

slip [m] after $\sigma_{xy} = .75\sigma_n$ with $\mu_s = 0.2$

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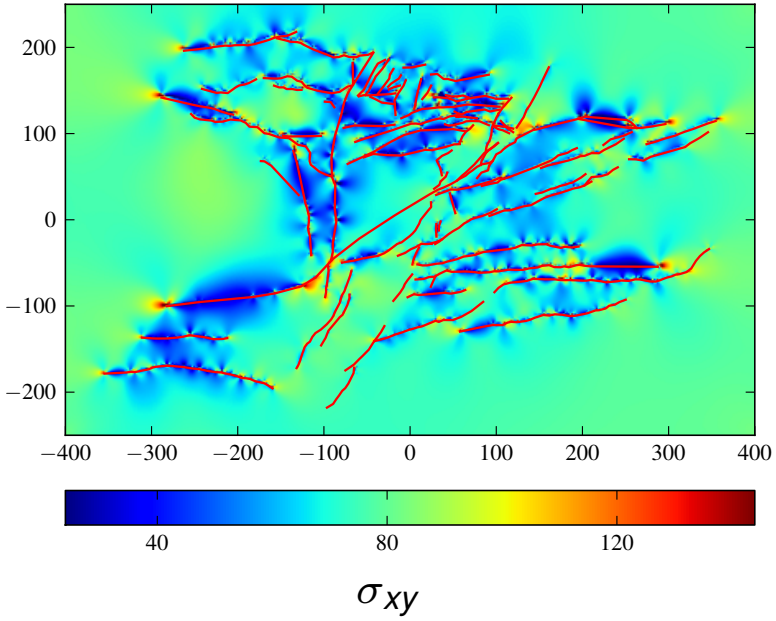

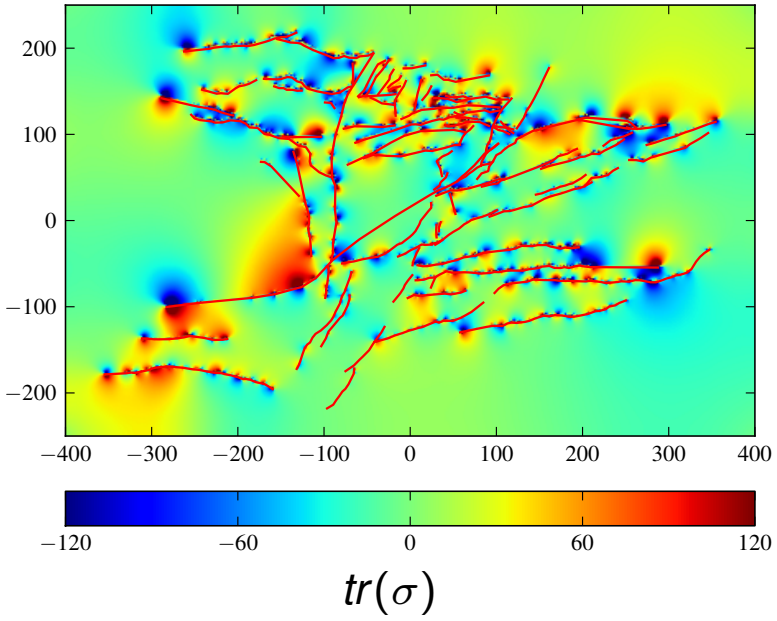

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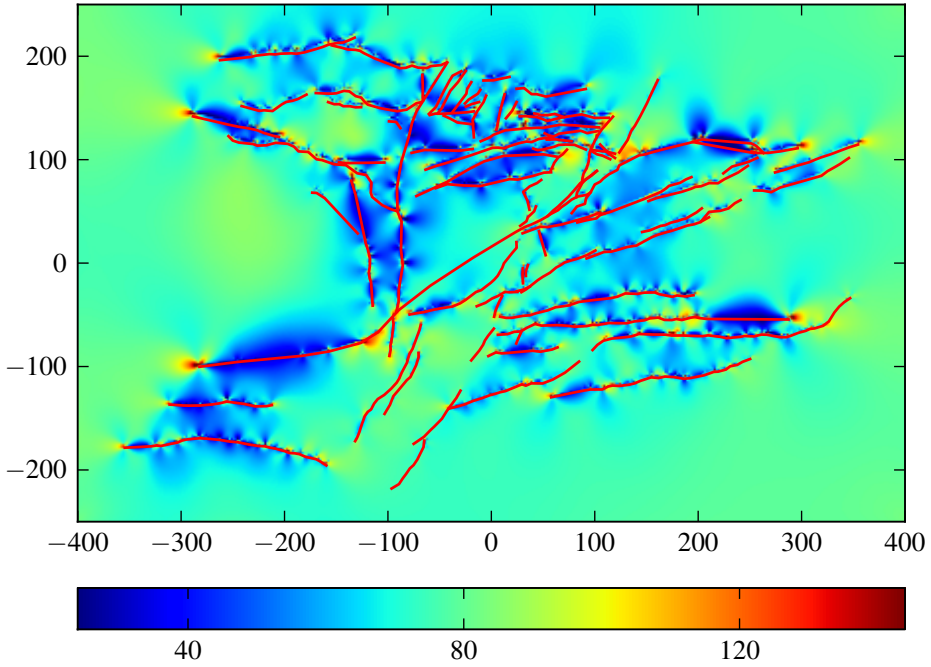
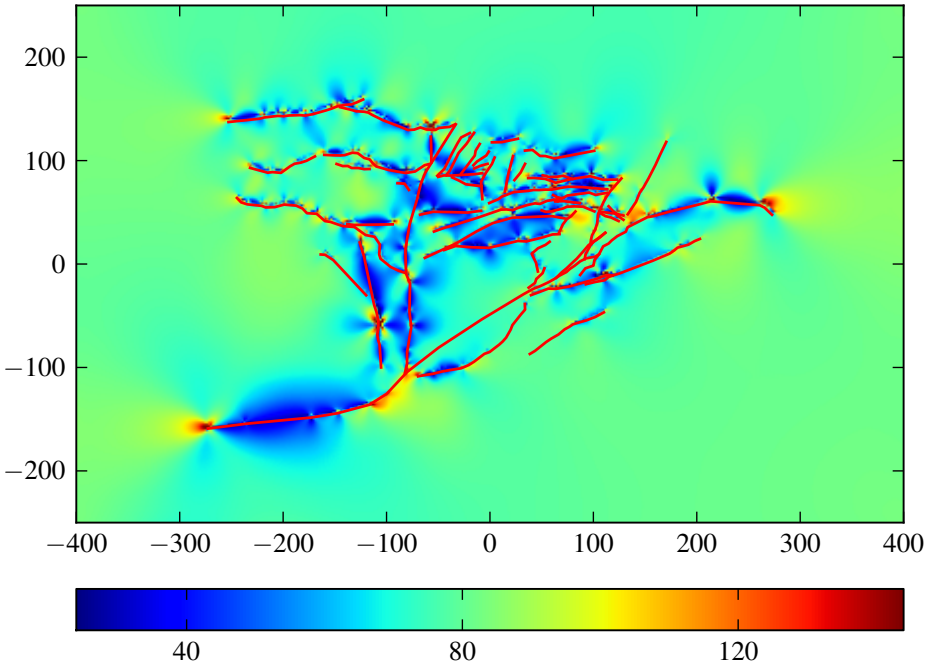
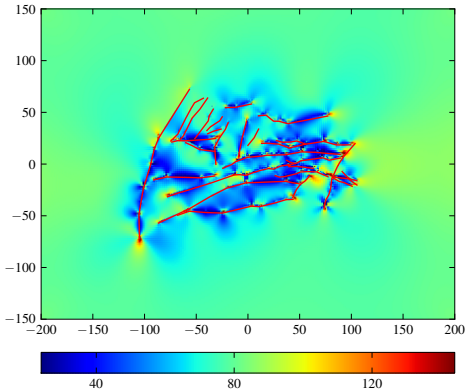






Conclusions

- Mesh generation is a hard problem, and is the primary impediment to system-scale models and simulation.
- FEM methods, even with a robust mesh generator, won't deal with all problems well (robustness to fault geometry, addition/subtraction/formation of faults)
- XFEM allows relaxation of mesh constraints on the FEM.
- In 2D, this enables solution of deformation in fault system simulations (with friction)

Discussion

Incorporation of XFEM in PyLith.

- Requires methods for dealing with fault mesh and background mesh interactions (discrete differential geometry, distance functions, etc) *Mesh generators do this now*
- Requires a fully weak formulation with stabilized, weak boundary conditions to deal with BB conditions while calculating tractions.

BB-condition

Incorporation of tractions and frictional failure requires a mixed formulation:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

where λ are tractions. Where the fault is stuck, this is the tractions required to keep the fault stuck. Where the fault is slipping, this is friction.

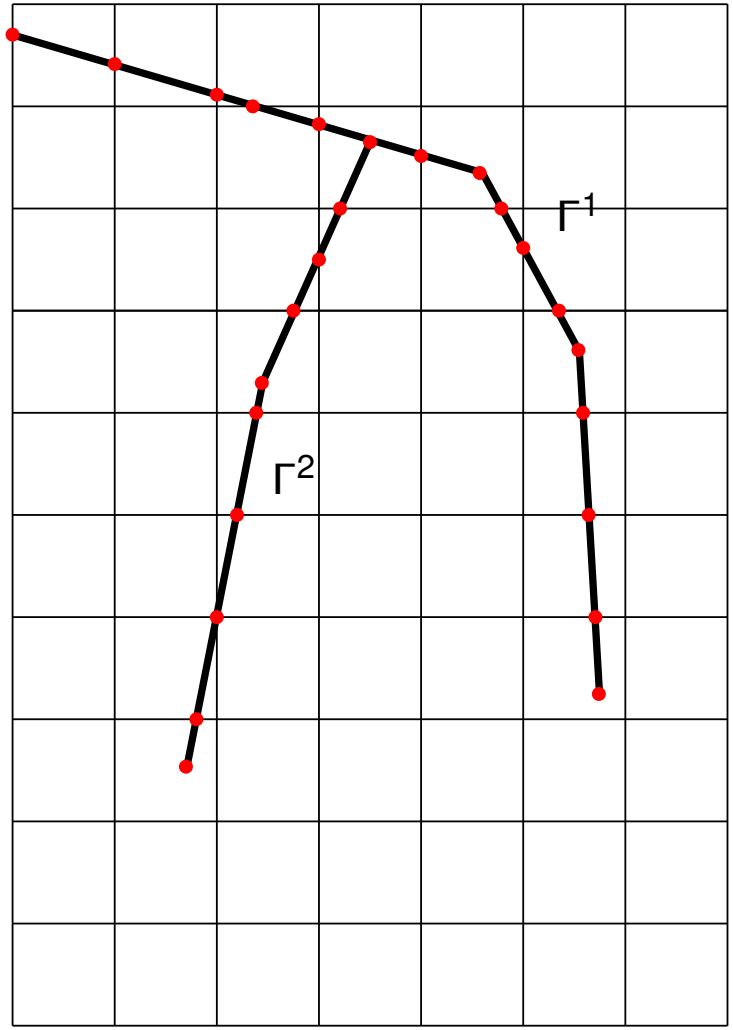
$u \in \mathcal{V}$, but we must choose a function space for λ, i.e. $\lambda \in \mathcal{W}$.

A compatibility condition, or Babuska-Brezzi (BB) condition limits the choices of this space. This condition requires that the bulk solution u not under-constrain the traction solution λ.

BB condition: if $B^T_{ij} = b(\psi_i, \lambda_j)$, require: $\inf_{w \in \mathcal{W}} \sup_{v \in \mathcal{V}} \frac{b(v,w)}{|v|\,|w|} > \epsilon > 0$

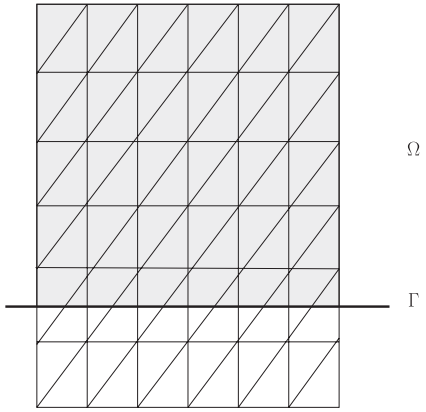
BB-condition

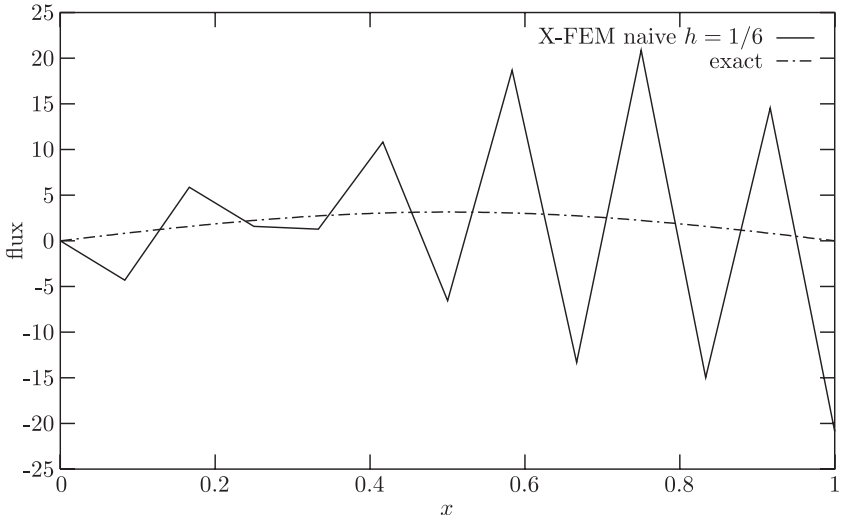
The obvious choice: choose piecewise linear functions on a fault mesh which consists of intersections of the fault with elements of the background mesh.

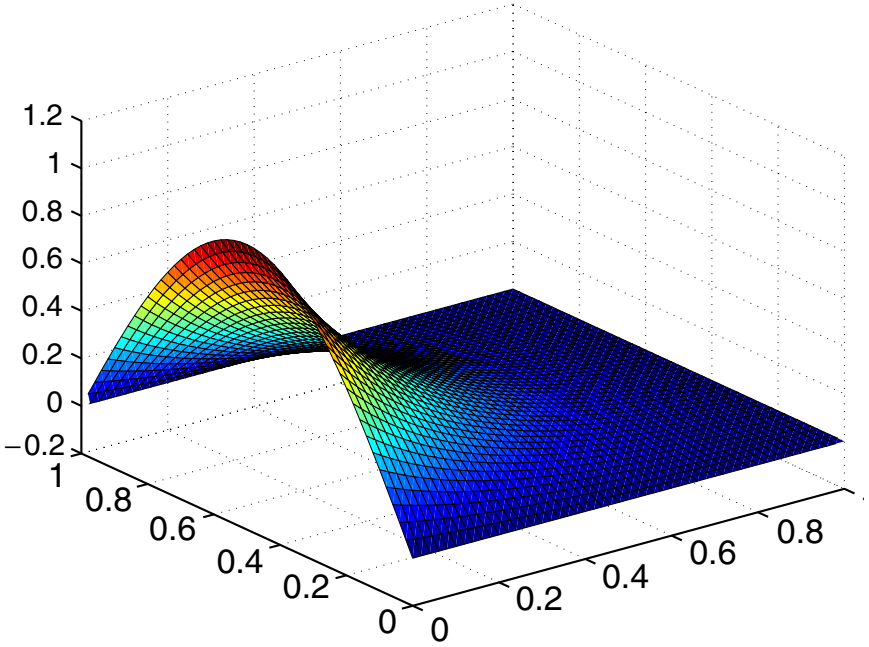


BB-condition

The obvious choice, in XFEM, is bad (Moes et al '06):

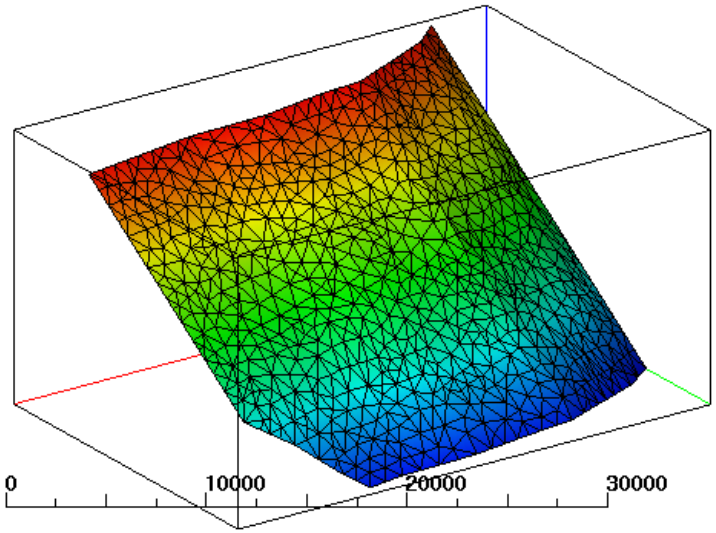


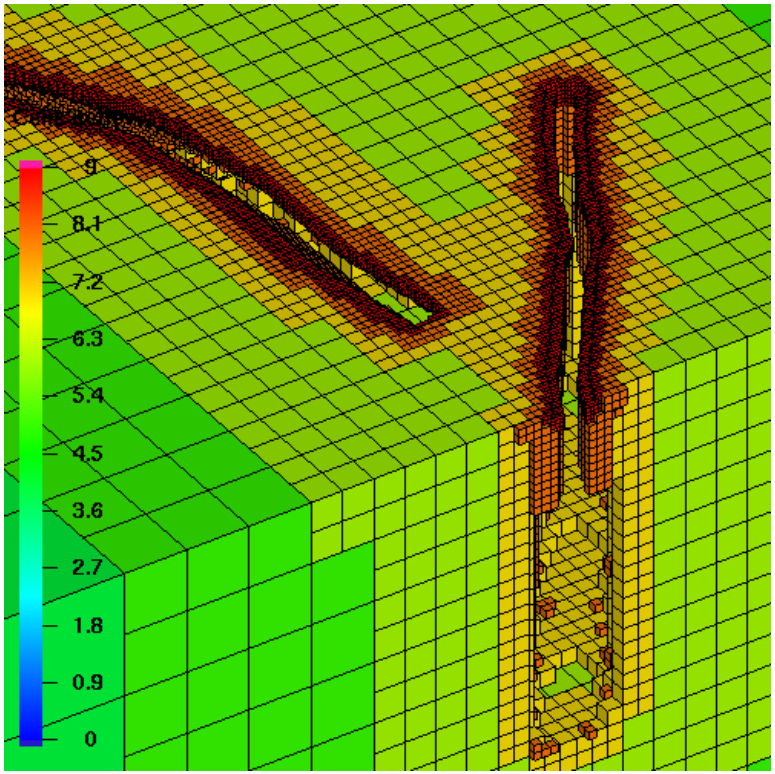




BB-condition

So, require $h^{\Gamma} > h^{\Omega}$





Conclusions

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- XFEM allows relaxation of mesh constraints on the FEM.
- In 2D, this enables solution of deformation in fault system simulations (with friction)
- Future work extends this to 3D, within PyLith. Much of the work needed is already done by PyLith and/or mesh libraries such as LaGrit. Compatibility conditions must be considered, but if $h^{\Gamma} > h^{\Omega}$ (a criteria easily enforced via mesh generators), simple function spaces may be BB-compatible.