Multilevel Approach for signal restoration problems with Toeplitz matrices

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Problem

Find x, given A, b and the model

$$Ax = b$$

where $b = b_{true} + e = Ax_{true} + e$.

Characteristics of the system:

- Very large and ill-conditioned matrix
- e is unknown (white) noise
- Decaying singular values without gap
- Singular vectors become more oscillatory

Background

Haar wavelet transform

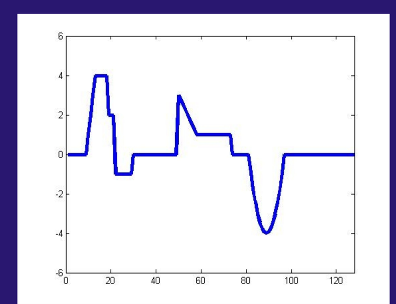
$$W^T = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} W_1^T \\ W_2^T \end{bmatrix}$$

In wavelet domain $Ax = b$ becomes

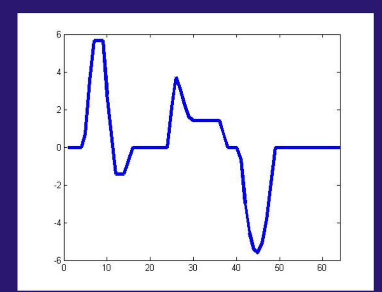
$$\hat{A}\hat{x} = \hat{b}$$

where $\hat{A} = W^T A W$, $\hat{x} = W^T x$ and $\hat{b} = W^T b$.

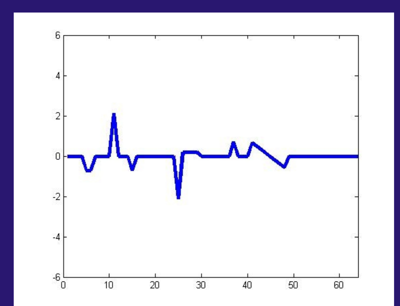
$$\begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix}$$



True Signal x



Scale coefficients x_1



Wavelet coefficients x_2

Multilevel Method

Pre-smoothing

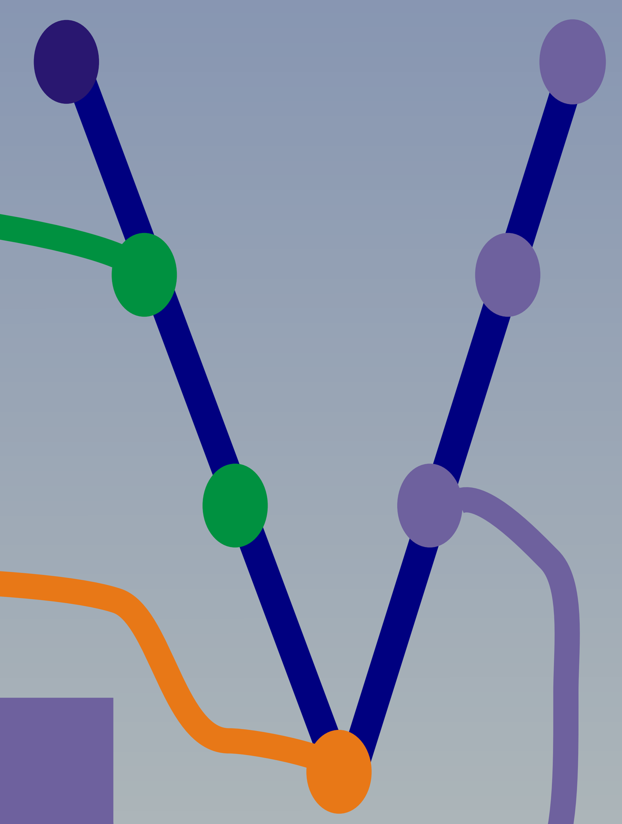
$$\hat{x}_1 = \text{LSQR}(\hat{A}_1, \hat{b}_1, 2 \text{ or } 3)$$

Coarse-grid Correction

$$\min_{\hat{x}_1}\left\{\left\|\hat{A}_1\hat{x}_1 - \hat{b}_1\right\|_2^2 + \lambda^p \left\|L\hat{x}_1\right\|_p^p\right\}$$

Post-smoothing

$$\min_{\hat{x}_2}\left\{\left\|\begin{bmatrix} \hat{A}_2 \\ \hat{A}_4 \end{bmatrix}\hat{x}_2 - \left(\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} - \begin{bmatrix} \hat{A}_1 \\ \hat{A}_3 \end{bmatrix}\hat{x}_1\right)\right\|_2^2 + \lambda^p \left\|L(x_{pre} + W_1\hat{x}_1 + W_2\hat{x}_2)\right\|_p^p\right\}$$



function $x^i = MGM(A^i, b^i)$

If coarsest grid

  $x^i = \text{Coarse-grid Correction}(A^i, b^i)$

Else

  If *non* finest grid

    $x^i = \text{Pre-smoothing}(A^i, b^i)$

  End If

  $r^i = b^i - A^i x^i$

  $b_{i+1} = W_1^T r_i; \quad A^{i+1} = W_1^T A^i W_1$

  $\hat{x}_1^{i+1} = MGM(A^{i+1}, b^{i+1})$

  $x^i_{new} = x^i + W_1 \hat{x}_1^{i+1}$

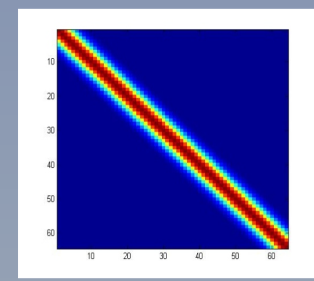
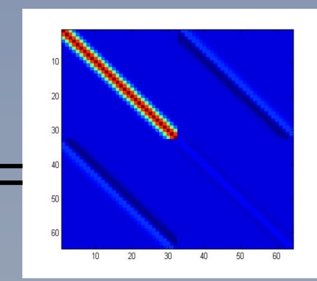
  $r^i_{new} = b^i - A^i x^i_{new}$

  $\hat{x}_2^{i+1} = \text{Post-smoothing}(WA^iW_2^T, Wr^i_{new}, L, x^i_{new})$

  $x^i = x^i_{new} + W_2 \hat{x}_2^{i+1}$

End If

Computational Issues

$A =$  $\quad \hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix} =$ 

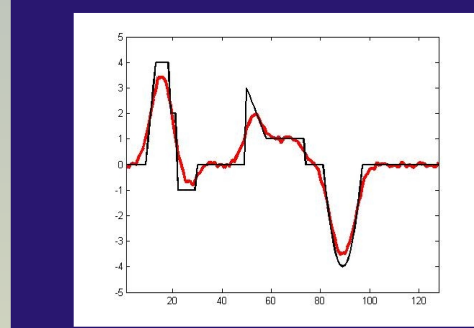
Theorem:

If A is Toeplitz, then $\hat{A}_1, \hat{A}_2, \hat{A}_3$ and $\hat{A}_4$ are Toeplitz.

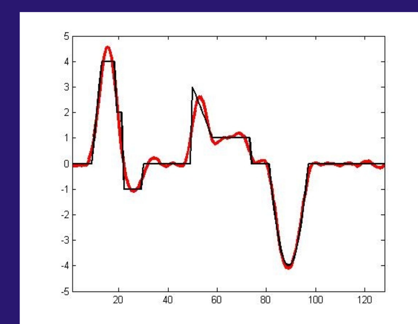
Consequences: Fast matrix - vector products.

No need to compute $\hat{A}_i$ explicitely.

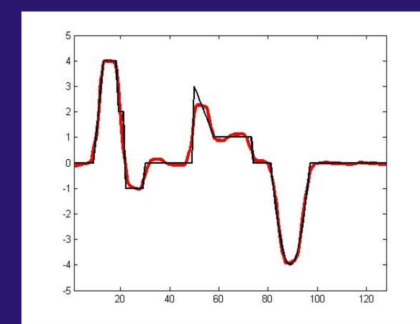
Results



Righ-hand side b



LSQR solution



MGM solution

Our new multilevel method captures the edges from the original signal. In contrast, the LSQR method returns smooth solutions.

References

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[2] P. C. Hansen. Rank-Deficient and Discrete Ill-posed Problems. SIAM, Philadelphia, 1998.

[3] V. E. Henson, W. L. Briggs and S. F. McCormick. A Multigrid Tutorial (2nd. Ed.). SIAM, Philadelphia, 2000.