Multiscale Finite Element Methods for Heterogeneous Porous Media

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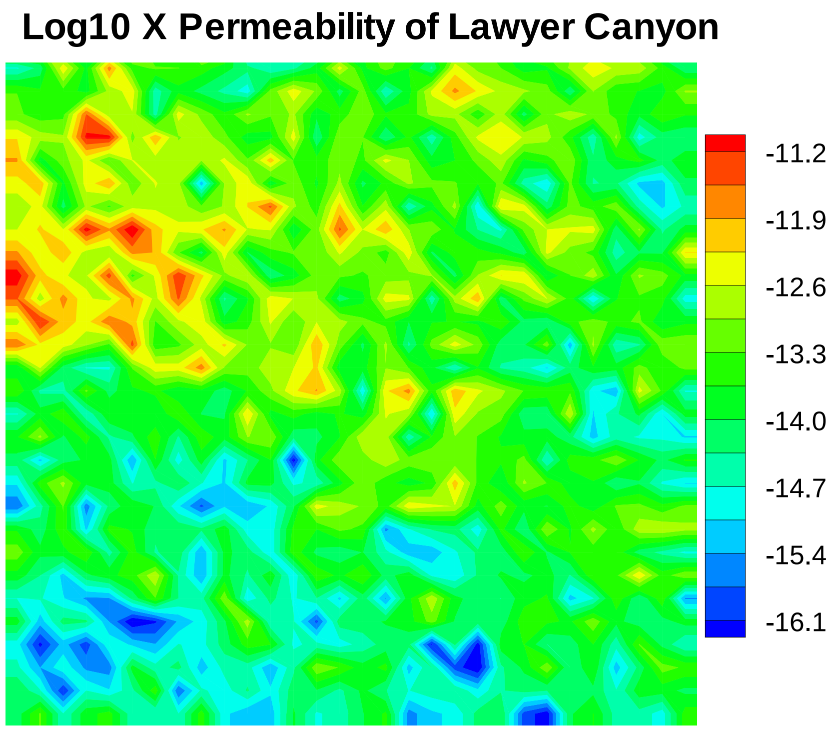
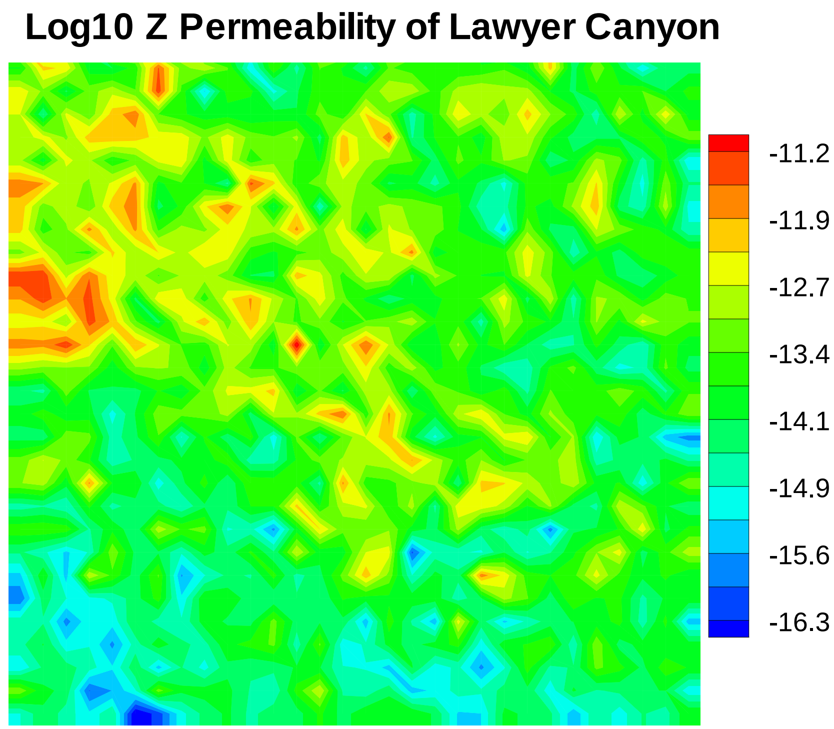
Outline

I. Heterogeneous Porous Media and Problems of Scale

Meter-Scale Natural Heterogeneity





Lawyer Canyon data, meter scale
(ranges by a factor of 10^6)

Difficulty: Fine-scale variation in the permeability K leads to fine-scale variation in the solution $(\mathbf{u}, p)$.

The Problem of Scale

Suppose K varies on the scale ϵ. Then

$$|\nabla p| = \mathcal{O}(\epsilon^{-1}) \qquad \text{and} \qquad |D^k p| = \mathcal{O}(\epsilon^{-k})$$

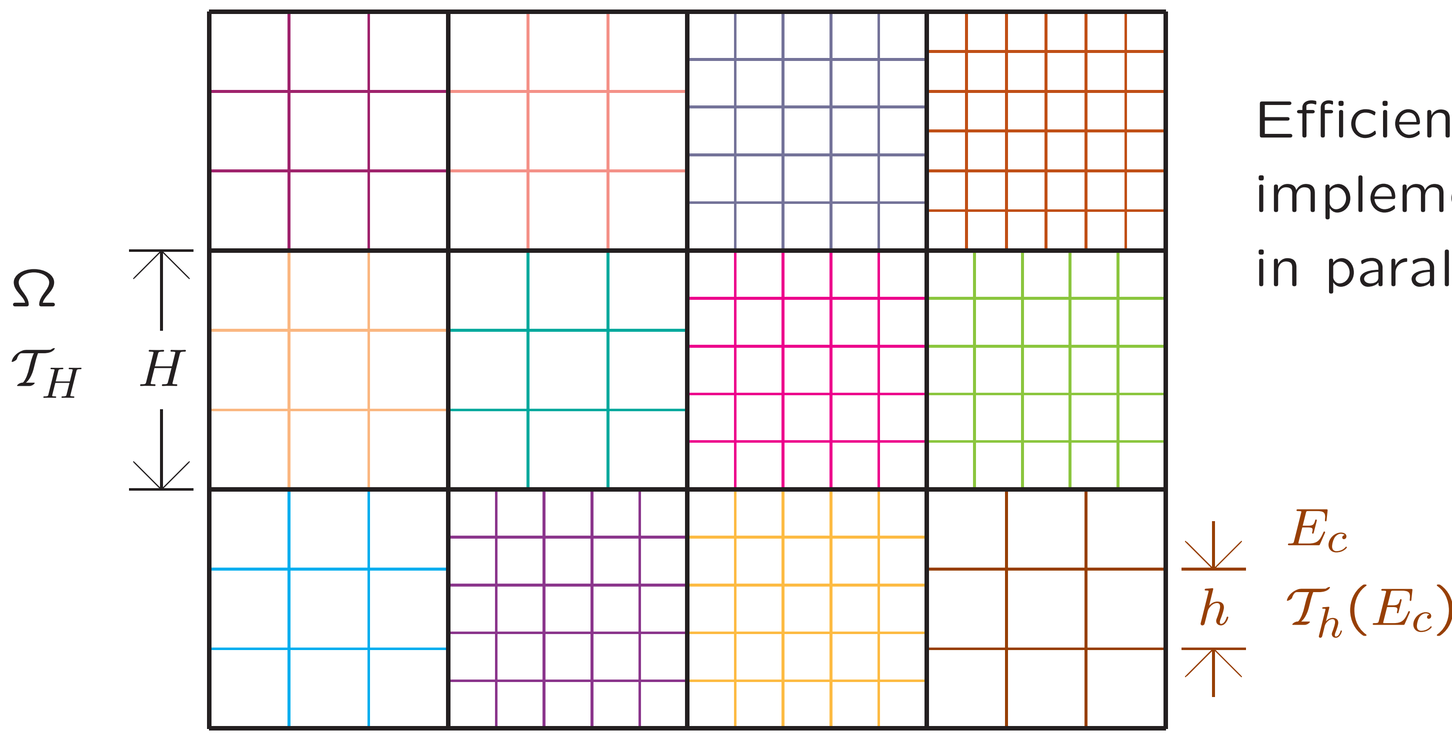
Typical error estimates. From polynomial approximation theory, the best approximation on a finite element partition $\mathcal{T}_h$ is

$$\inf_{q \in \mathbb{P}_{k-1}(\mathcal{T}_h)} \|p - q\|_0 \leq C\|p\|_k h^k \sim C\left(\frac{h}{\epsilon}\right)^k$$

- To resolve p in a standard approximation, we need a grid size $h < \epsilon$. That is, we must resolve K.
- If $h > \epsilon$, this is *not* small, and we need a new approach.

Overall Multiscale Strategy—1

1. **Localization.** The full PDE problem is decomposed into many small, local, coarse element subproblems (of scale $H > \epsilon$).
2. **Fine-scale effects.** The local subproblems are given appropriate boundary conditions and solved on the fine scale $h < \epsilon$ (resolving variations in K) to define a coarse scale finite element basis.
3. **Global coarse-grid problem.** This H-scale coarse basis is used to approximate the solution globally.
4. **Fine-grid construction.** The finite element basis encapsulates an h-scale fine representation of the solution.



Efficiently implemented in parallel!

Overall Multiscale Strategy—2

- The problem is fully resolved on the fine scale.
- The problem is *not* fully coupled. The global problem is a reduced degree-of-freedom system.
- Computational efficiency comes from divide-and-conquer:
  (a) Small, localized subproblems are easily solved (in parallel);
  (b) The coupled global problem involves far fewer degrees of freedom than the full fine-grid system (a few per coarse element), and so is relatively easily solved.

Some Multiscale Approaches

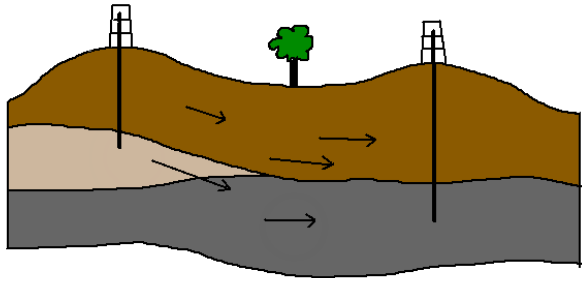
Variants of this strategy (Sorry, this is a very incomplete list!)

- **Generalized finite elements**
  1. Babuška, Caloz & Osborn 1994
  2. Stroubolis, Babuška & Copps 2001
- **Variational multiscale analysis**
  1. Hughes 1995
  2. Hughes, Feijóo, Mazzei & Quincy 1998
  3. Arbogast, Minkoff & Keenan 1998
  4. Brezzi 1999
  5. Arbogast 2004
  6. Arbogast & Boyd 2006
- **Multiscale finite elements**
  1. Hou & Wu 1997
  2. Hou, Wu & Cai 1999
  3. Efendiev, Hou & Wu 2000
  4. Chen & Hou 2003
  5. Aarnes 2004
  6. Aarnes, Krogstad & Lie 2006
- **Multiscale finite volumes**
  1. Jenny, Lee & Tchelepi 2003
  2. He & Ren 2004
  3. Ginting 2004
  4. Hesse, Mallison & Tchelepi 2008
- **Multiscale multigrid methods**
  1. Moulton, Dendy & Hyman 1998
- **Heterogeneous multiscale methods**
  1. E & Engquist 2003
- **Multiscale basis optimization**
  1. Rath 2007 (Ph.D. dissertation)
- **Multiscale mortar methods**
  1. Arbogast, Pencheva, Wheeler & Yotov 2007

II. Mixed Multiscale Finite Elements

(with Boyd and Rath)

Second Order Elliptic PDE'S in Mixed Form



Incompressible, single phase flow in a porous medium:

$$\begin{cases} \mathbf{u} = -K\nabla p & \text{in } \Omega & \text{(Darcy's law)} \\ \nabla\cdot\mathbf{u} = f & \text{in } \Omega & \text{(conservation)} \\ \mathbf{u}\cdot\nu = 0 & \text{on } \partial\Omega & \text{(BC for simplicity)} \end{cases}$$

A mixed variational formulation:
Find $p \in W = L^2$ and $\mathbf{u} \in \mathbf{V} = H(\text{div})$ such that

$$(K^{-1}\mathbf{u},\mathbf{v}) = -(\nabla p,\mathbf{v}) = (p,\nabla\cdot\mathbf{v}) \quad \forall\ \mathbf{v}\in\mathbf{V} \quad \text{(Darcy's law)}$$

$$(\nabla\cdot\mathbf{u},w) = (f,w) \quad \forall\ w\in W \quad \text{(conservation)}$$

Finite elements: Solve over finite dimensional spaces $\mathbf{V}_h \times W_h \subset \mathbf{V}\times W$.

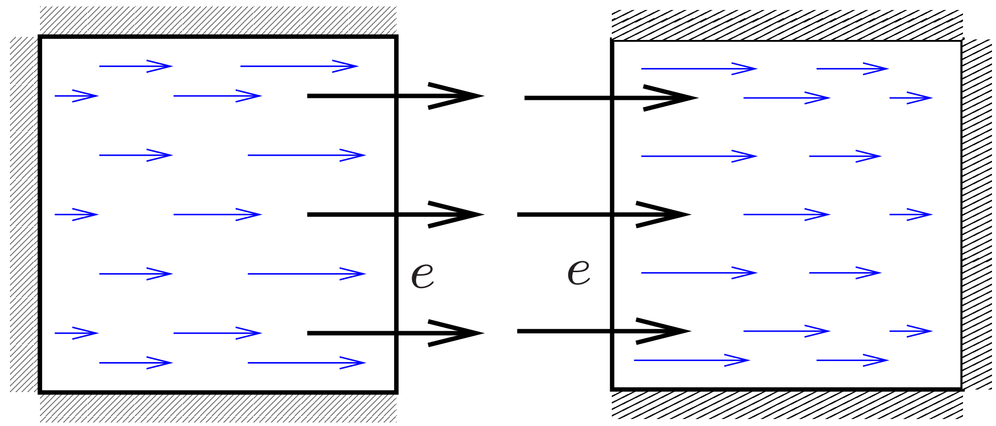
$$W_h = \text{ piecewise discontinuous constants}$$

The Velocity Mixed Multiscale Finite Elements

We define V_h on a coarse element E with edge e.

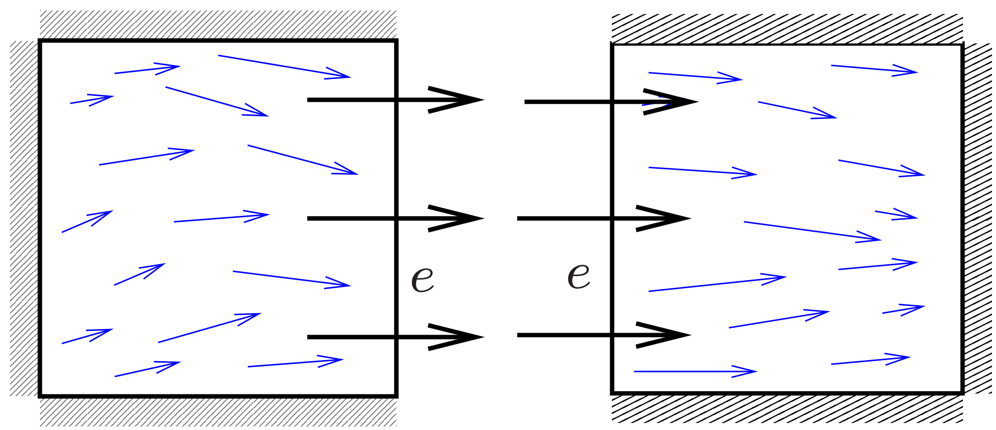
Standard Raviart-Thomas (RT0) finite element.

$$\begin{cases} R_e = -\nabla\omega \\ \nabla\cdot R_e = 1/|E| \\ R_e\cdot\nu = \begin{cases} 1/|e| & \text{on } e \\ 0 & \text{otherwise} \end{cases} \end{cases}$$



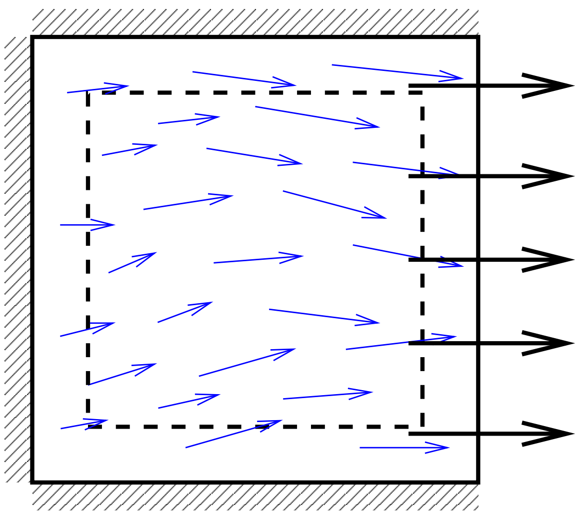
Variational multiscale finite element:

$$\begin{cases} R_e^{\mathsf{MS}} = -K\nabla\omega \\ \nabla\cdot R_e^{\mathsf{MS}} = 1/|E| \\ R_e^{\mathsf{MS}}\cdot\nu = \begin{cases} 1/|e| & \text{on } e \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

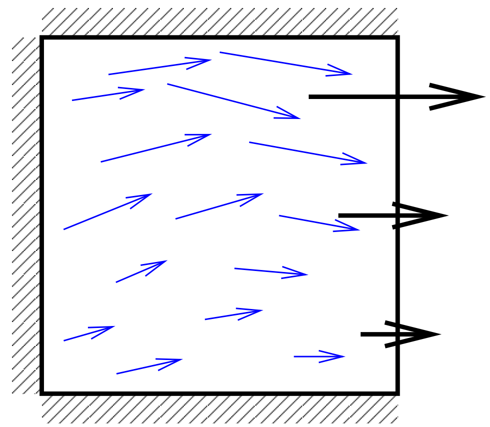


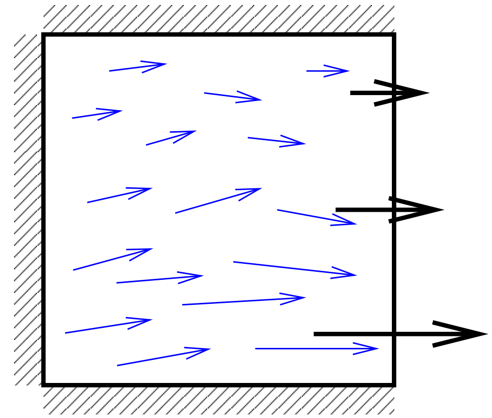
Boundary Conditions for Local Subproblems

Neumann BCs for constant outflow, but oversample. (Hou et al., 1997, 2003) Results in a nonconforming method.

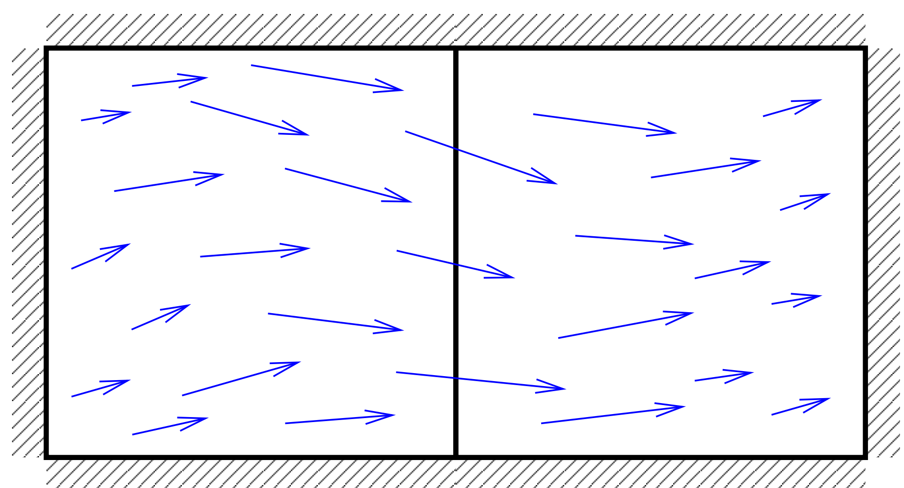


Neumann BCs for linear outflow. (Arbogast, 2000)





Dual element problem with source and sink terms. (Aarnes et al., 2004)

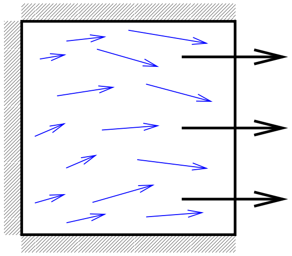


Estimates of the Pressure and Velocity Errors

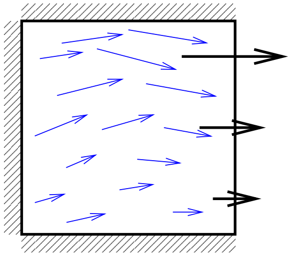
Error Estimates from Polynomial Approximation Theory

Let L be the order of approximation of the coarse mixed finite element velocity space used. Typically:

L=1 for lowest order Raviart-Thomas (RT0) spaces



L=2 for lowest order Brezzi-Douglas-Marini (BDM1) spaces



Theorem (A., 2004).

$$\|\mathbf{u} - \mathbf{u}_H\|_0 \leq C\|\mathbf{u}\|_L H^L = \mathcal{O}\left(\left(\frac{H}{\epsilon}\right)^L\right)$$

$$\nabla \cdot \mathbf{u}_H = f$$

$$\|p - p_H\|_0 \leq C\|\mathbf{u}\|_L H^{L+1} = \mathcal{O}\left(\left(\frac{H}{\epsilon}\right)^L H\right)$$

Suppose that K is locally periodic of period ϵ. Then

$$K(x) = \kappa(x, x/\epsilon)$$

where $\kappa(x, y)$ is periodic in y of period 1 on the unit cube Y.

Let K_0 be the homogenized permeability matrix, defined by

$$K_{0,ij}(x) = \int_Y \kappa(x,y)\left(\delta_{ij} + \frac{\partial \omega_j(x,y)}{\partial y_i}\right) dy$$

where, for fixed x, $\omega_j(x, y)$ is the Y-periodic solution of

$$-\nabla_y \cdot (\kappa \nabla_y \omega_j) = \frac{\partial \kappa}{\partial y_j}$$

Homogenized solution: Let $(\mathbf{u}_0, p_0)$ solve

$$\begin{cases} \mathbf{u}_0 = -K_0 \nabla p_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_0 = f & \text{in } \Omega \\ \mathbf{u}_0 \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

Then $(\mathbf{u}_0, p_0)$ is a smooth "approximation" of $(\mathbf{u}, p)$.

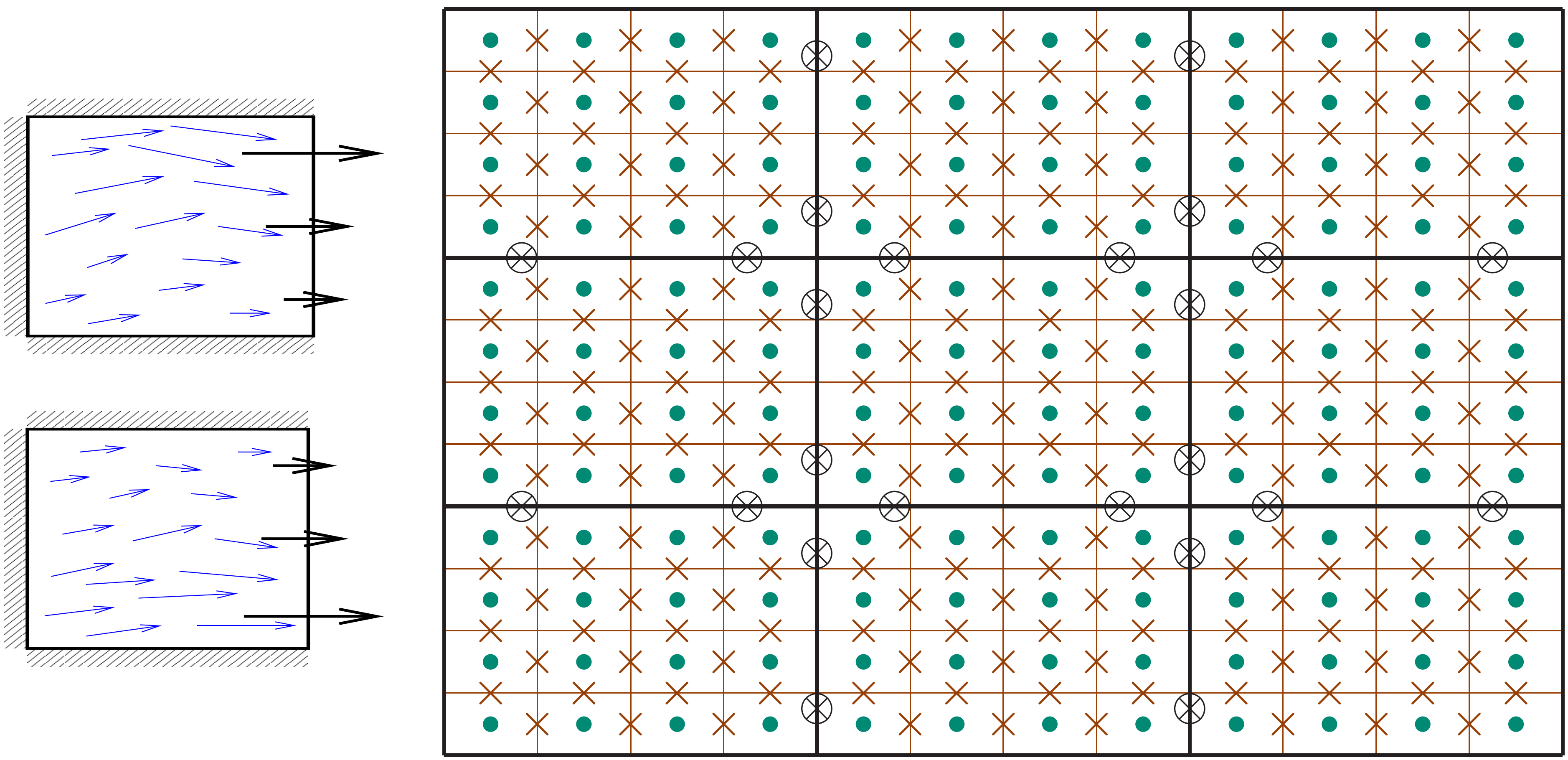
Multiscale Error Estimates

Theorem (Chen and Hou 2003; A. and Boyd, 2005). Assuming periodicity and the mixed variational multiscale method with $L = 1$ (RT0) or 2 (BDM1):

$$\|\mathbf{u} - \mathbf{u}_H\|_0 \le C\left\{\epsilon\|p_0\|_2 + \sqrt{\frac{\epsilon}{H}}\|p_0\|_{1,\infty} + H^L\Big(\|\mathbf{u}_0\|_L + \|f\|_{L-1}\Big)\right\}$$
$$= \mathcal{O}(H^L + \sqrt{\epsilon/H})$$
$$\|p - p_H\|_0 \le C\Big(\epsilon + (\epsilon/H)^{1/d-\eta} + H\Big)\|\mathbf{u} - \mathbf{u}_H\|_0 \quad \text{(superconvergence)}$$

where d is the space dimension and $\eta > 0$ if $d = 2$ and $\eta = 0$ if $d = 3$.

Composite Numerical Grid for BDM1–RT0



⊗ Coarse velocity (linear) × Subgrid velocity

• Pressure

We fully resolve K, but only partially couple the dynamics.

Numerical Examples and Application to Subsurface Flow Simulation

Application to Waterflood Simulation

Use standard equations and sequential solution.

Pressure equation: Global pressure formulation.

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{u} = q(P)$$
$$\mathbf{u} = -K\lambda(S)\Big(\nabla P - \rho(S) g \mathbf{e}_3\Big)$$

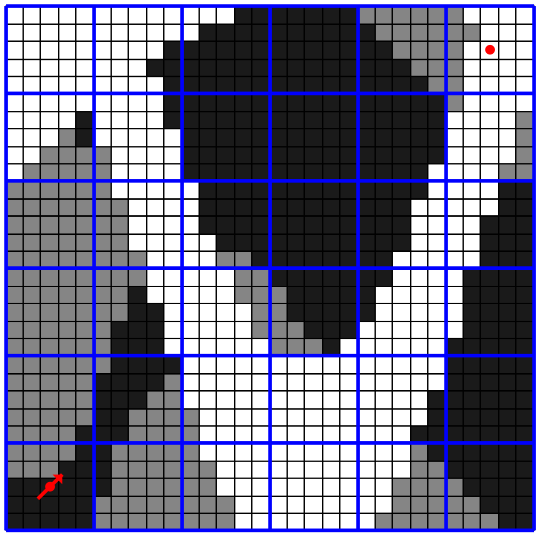
Upscale this equation. Use BDM1/RT0 unless otherwise noted.

Saturation equation: Kirchhoff formulation.

$$\frac{\partial \phi S}{\partial t} + \nabla \cdot \mathbf{u}_w = q_w(S)$$
$$\mathbf{u}_w = -K\nabla Q(S) + c(\mathbf{u}, S)$$

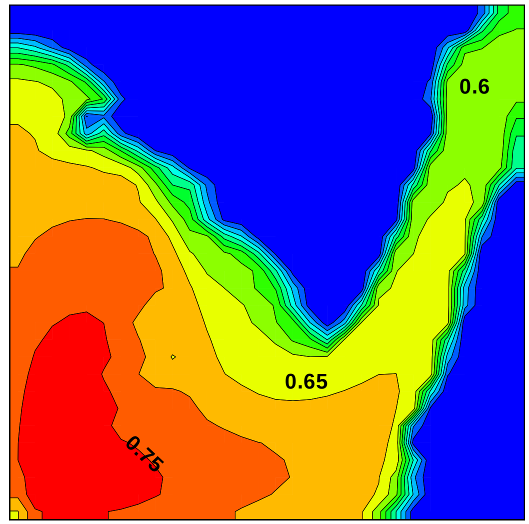
Solve on the fine scale.

A Fluvial Subsurface Environment–1

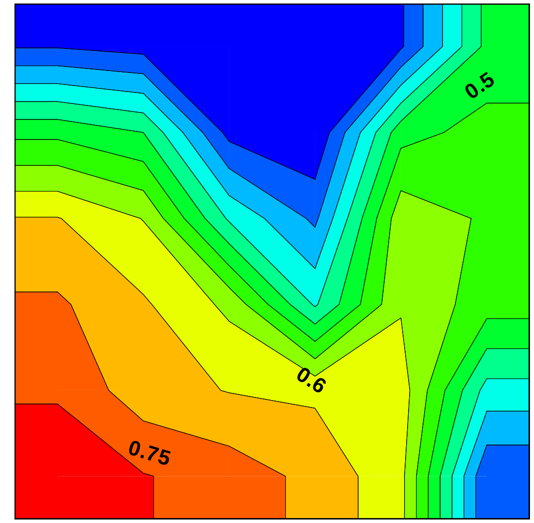


K = 0.1 D
K = 1.0 D
K = 10.0 D

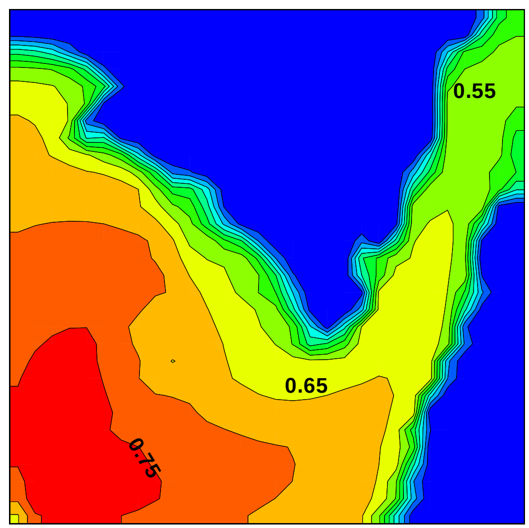
Permeability field
(White & Horne, 1987)



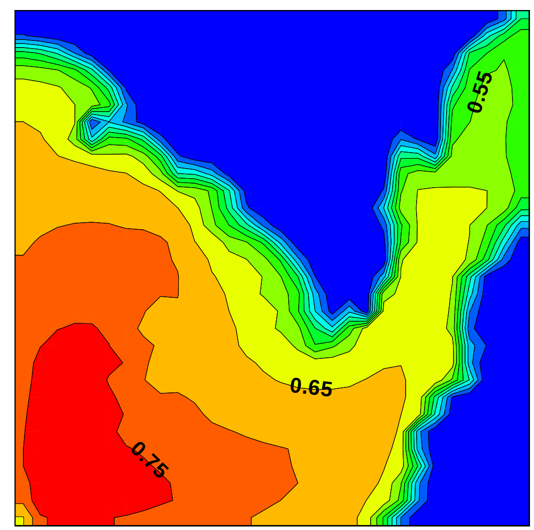
Fine 30 × 30



Average K 6 × 6



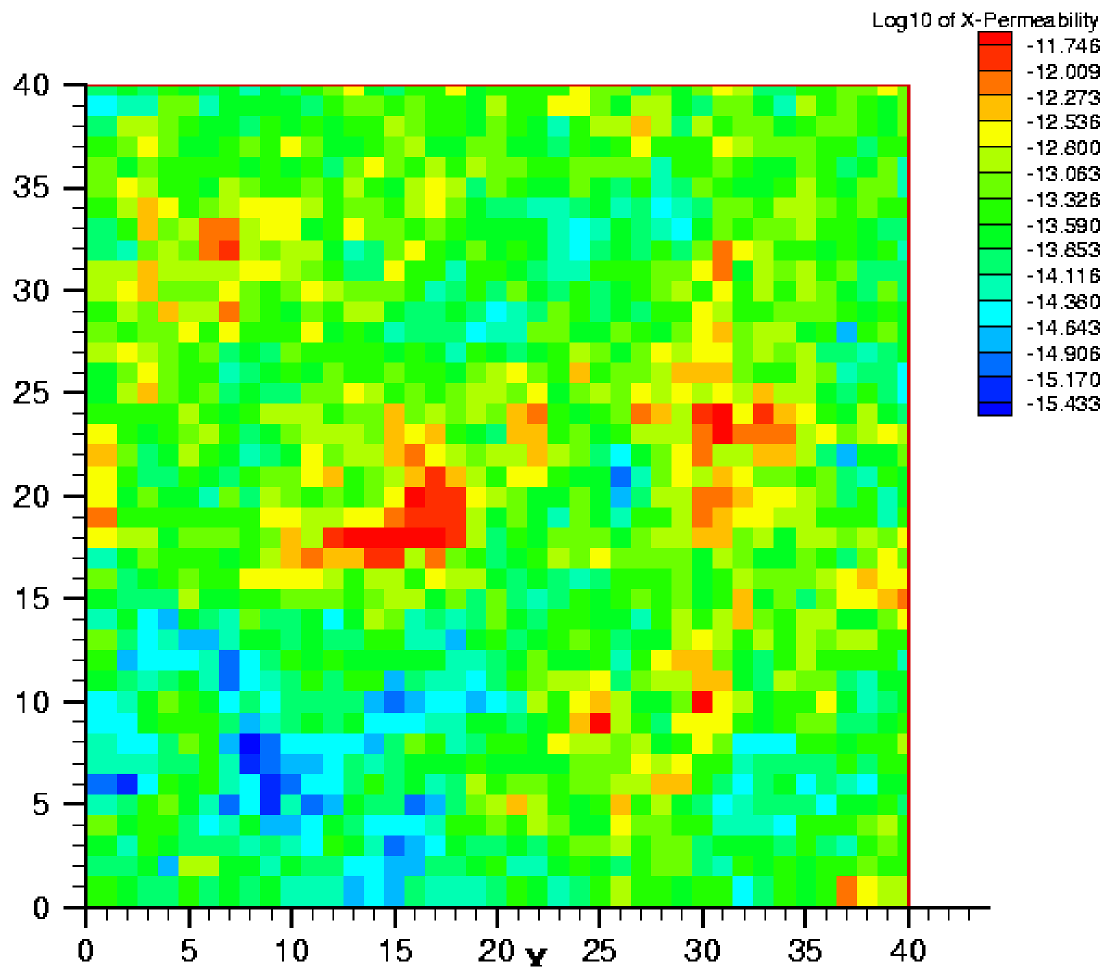
Upscaled to 6 × 6



Upscaled to 3 × 3

A Quarter Five-spot Oil Reservoir Waterflood—1

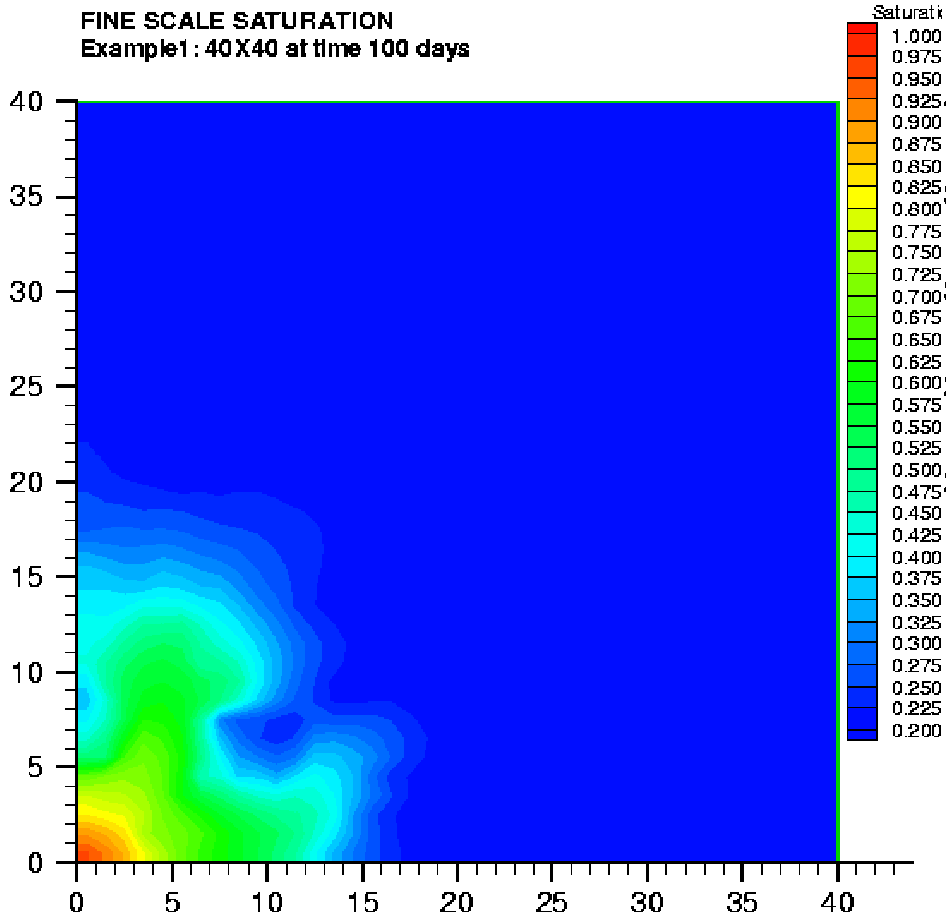
Logarithm of the permeability



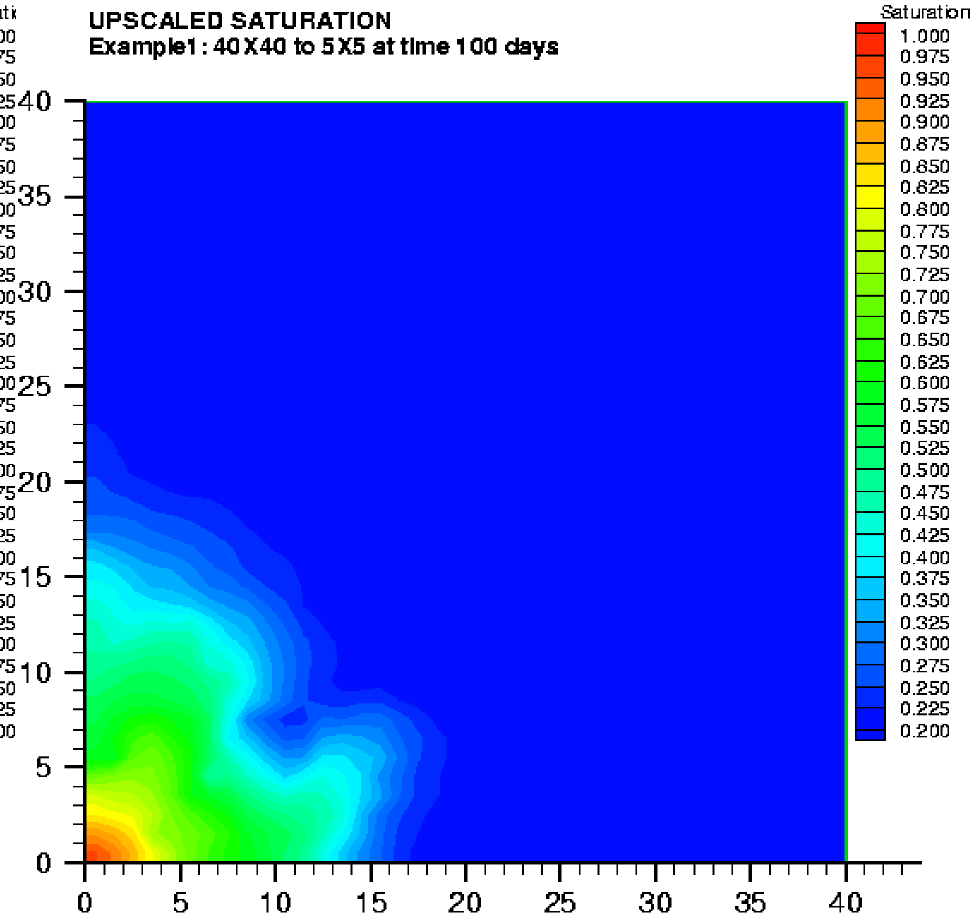
Fine 40×40

A Quarter Five-spot Oil Reservoir Waterflood—2

Water saturation contours at 100 days



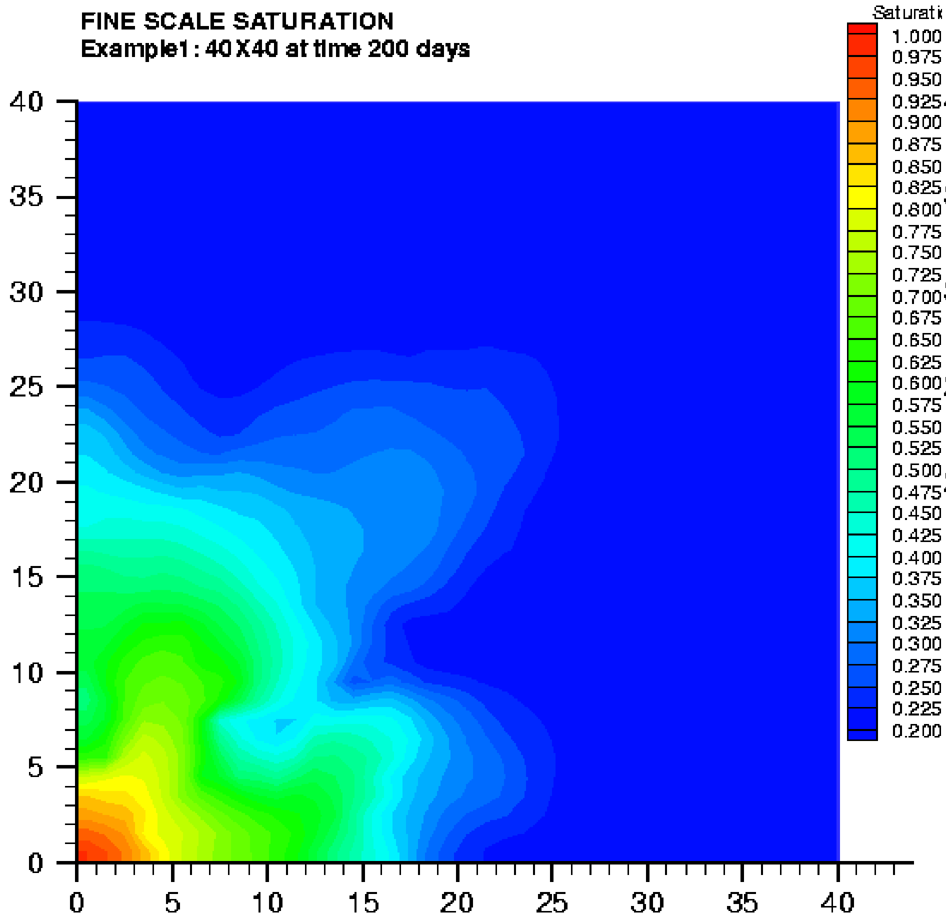
Fine 40×40



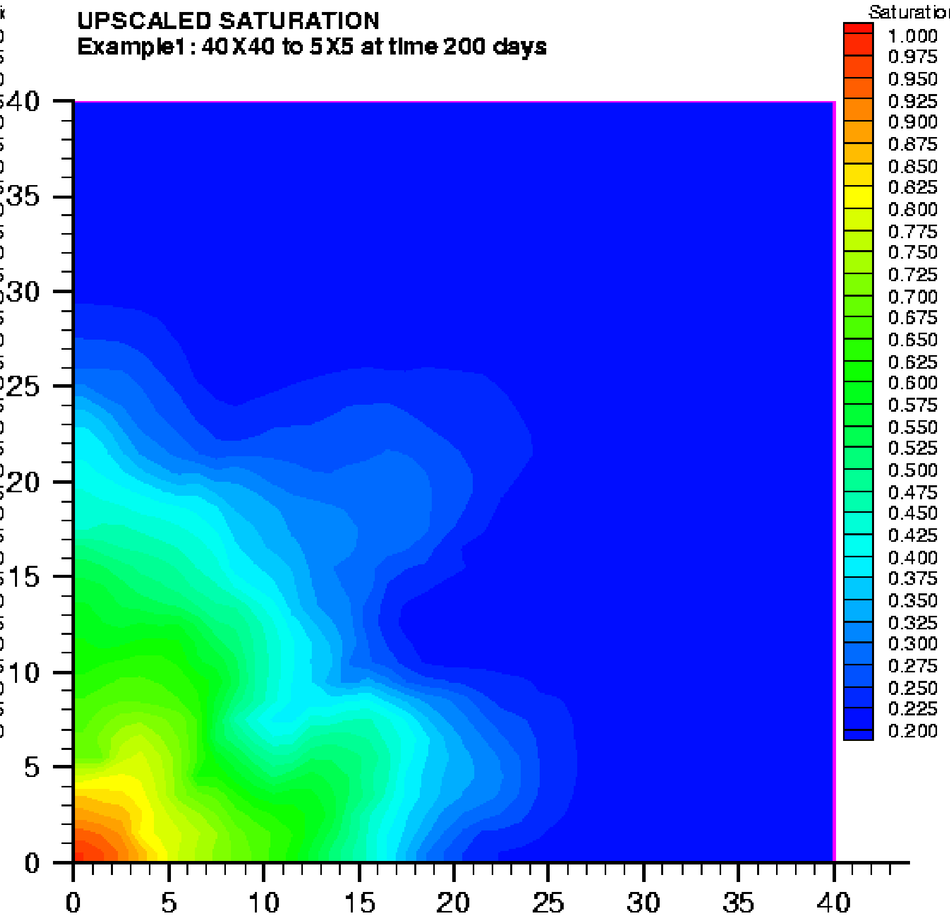
Fully upscaled to 5×5

A Quarter Five-spot Oil Reservoir Waterflood—3

Water saturation contours at 200 days



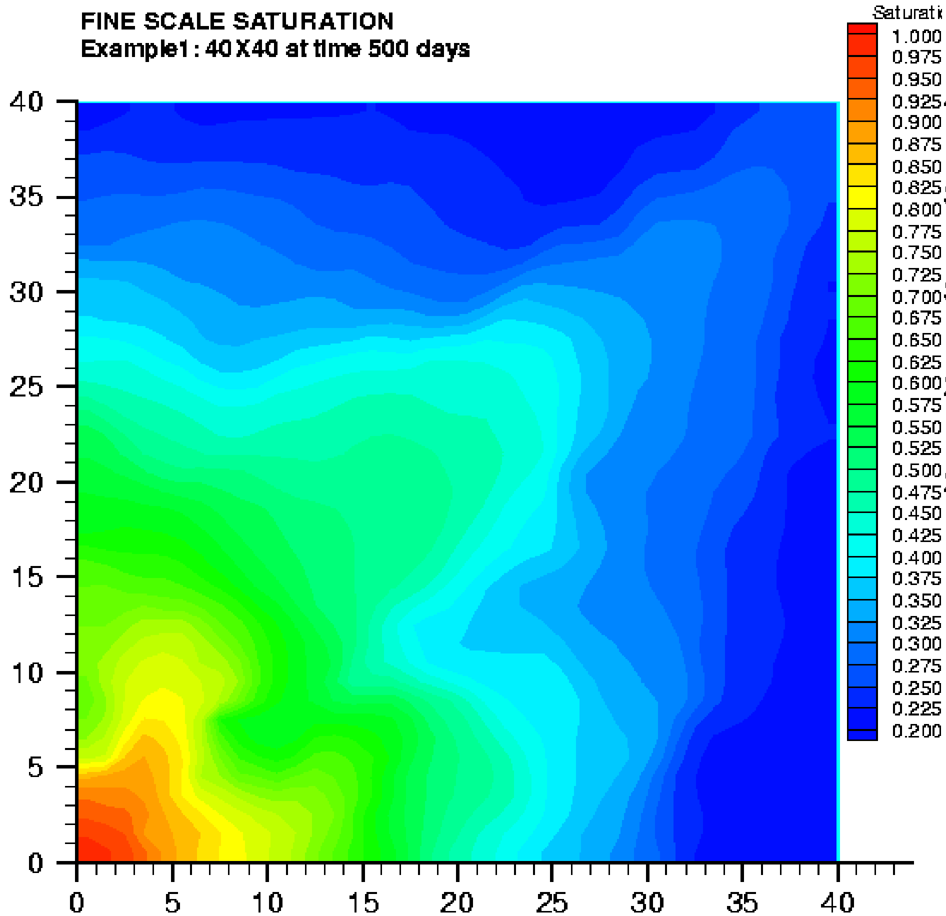
Fine 40 × 40



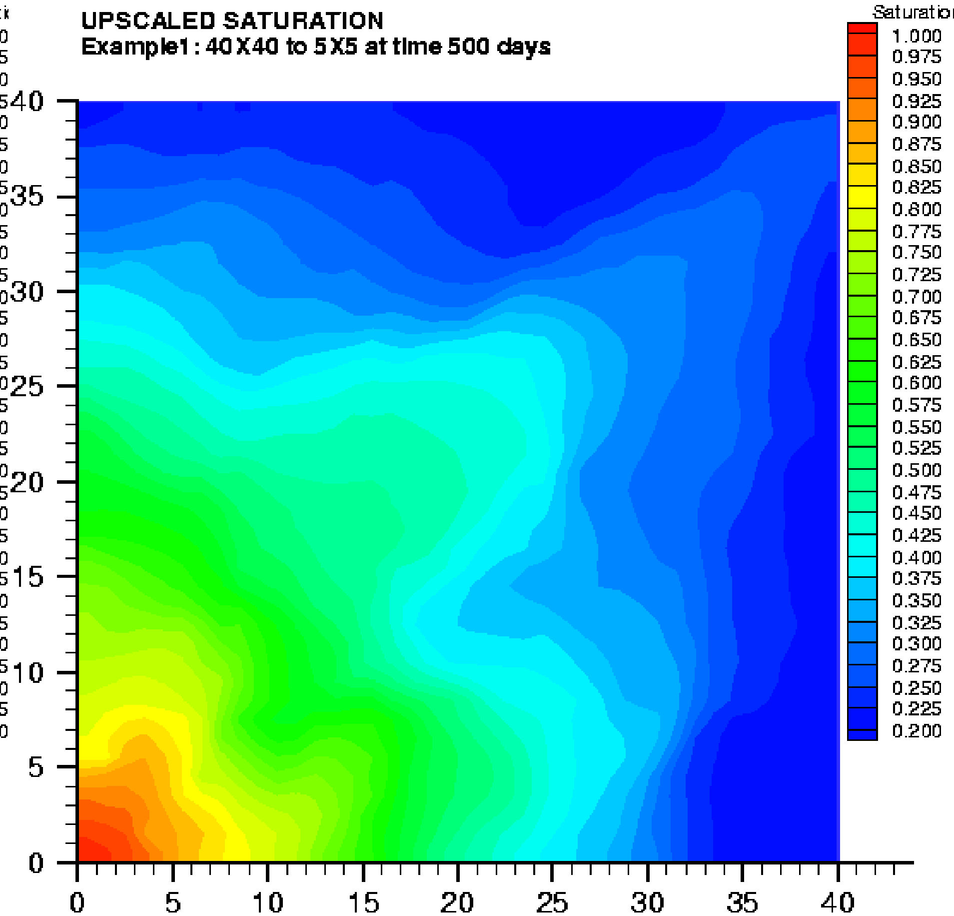
Fully upscaled to 5 × 5

A Quarter Five-spot Oil Reservoir Waterflood—4

Water saturation contours at 500 days



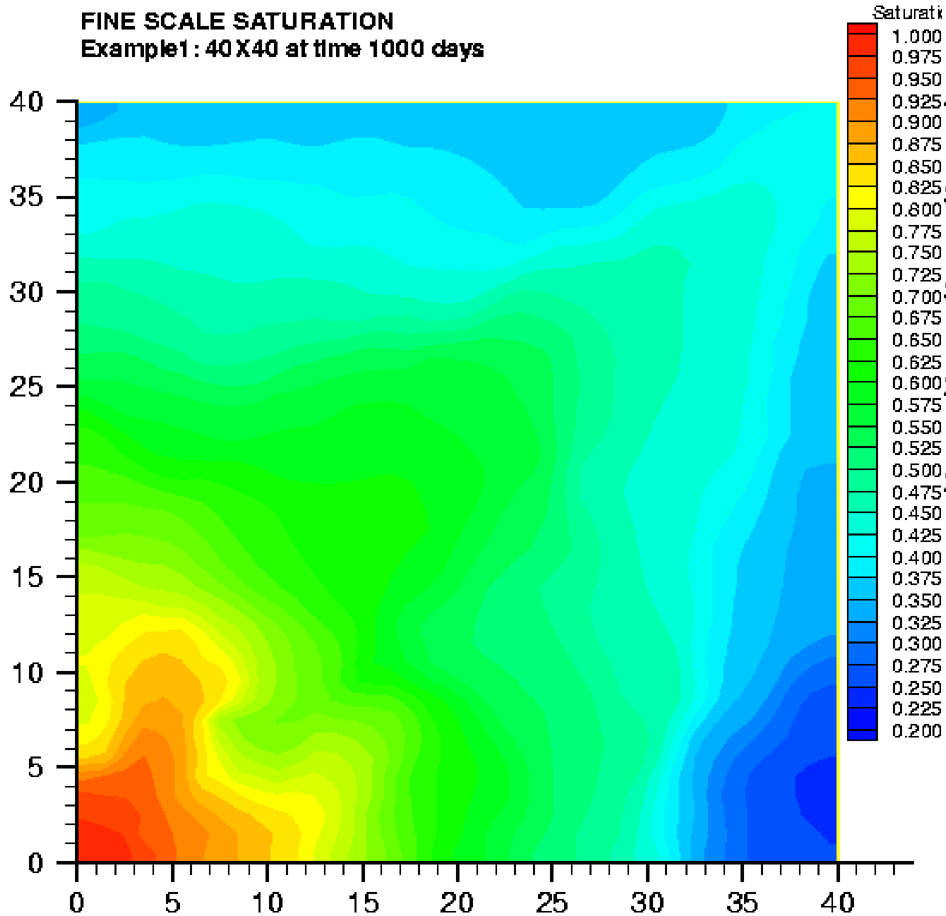
Fine 40 × 40



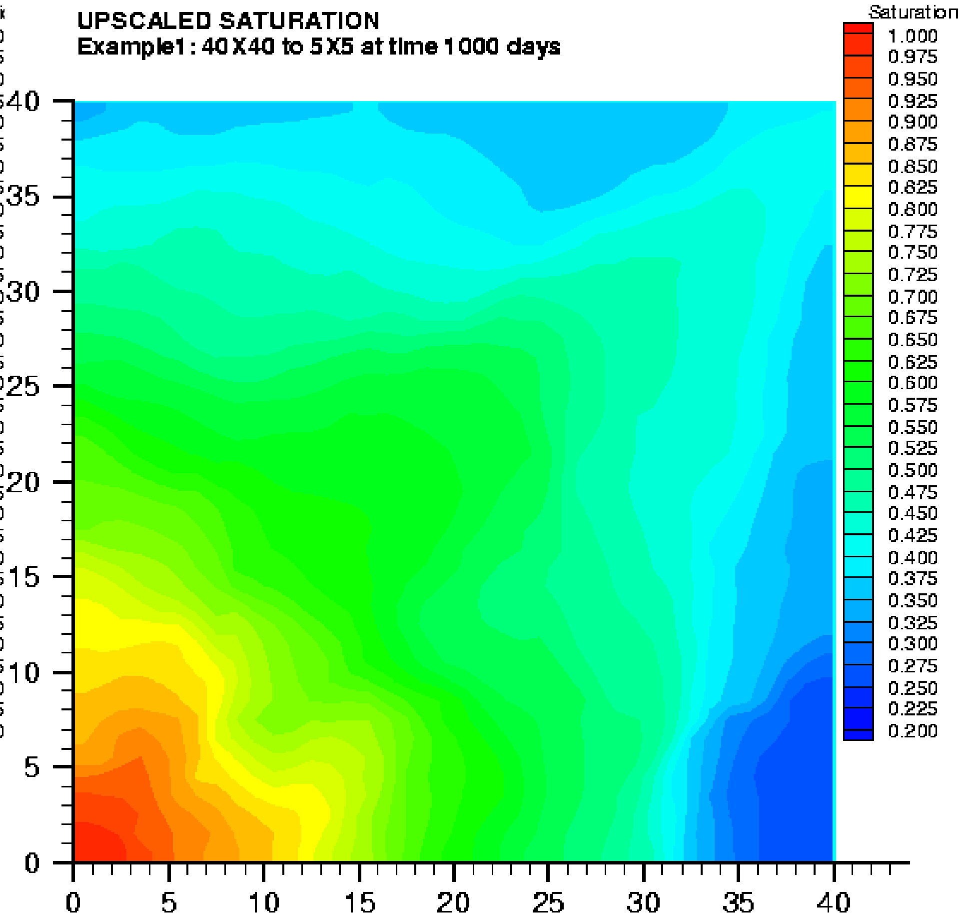
Fully upscaled to 5 × 5

A Quarter Five-spot Oil Reservoir Waterflood—5

Water saturation contours at 1000 days



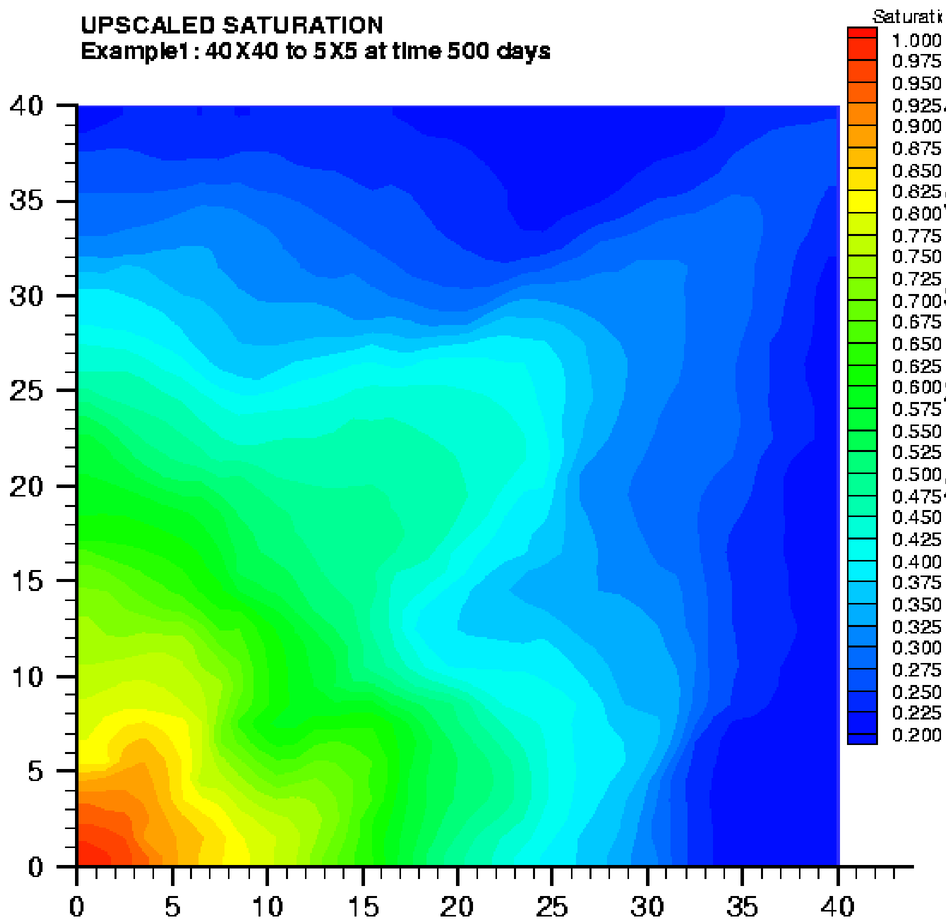
Fine 40×40



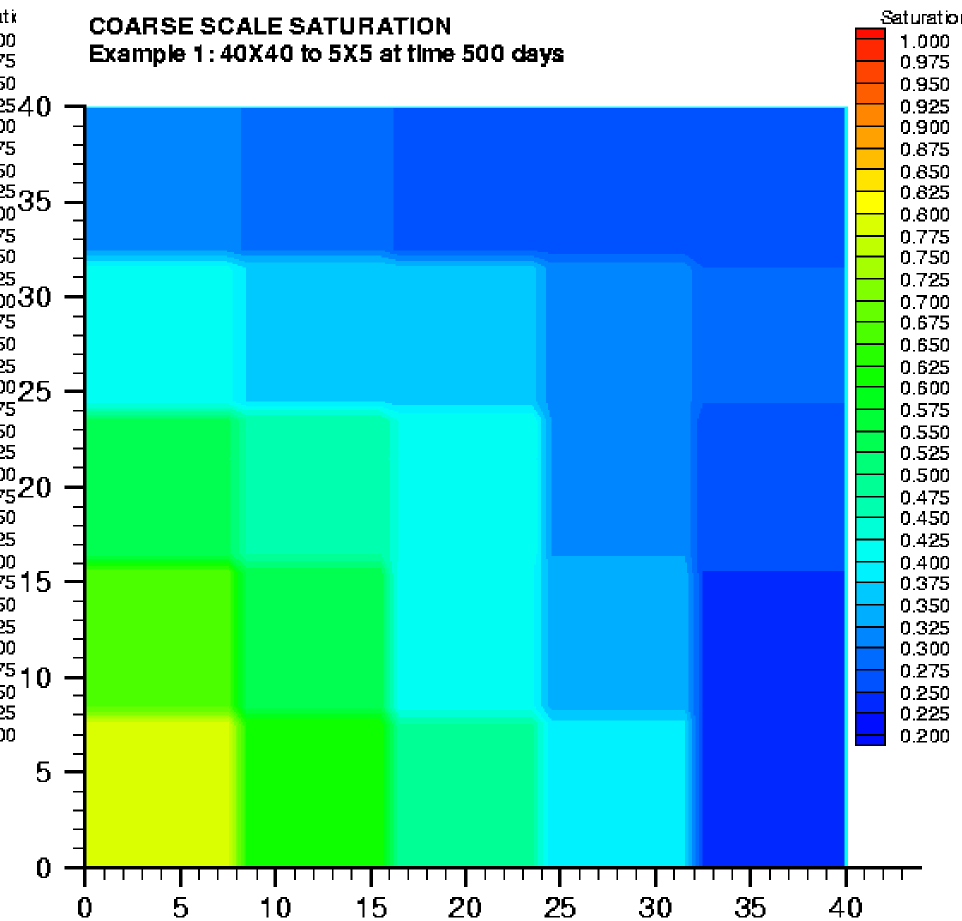
Fully upscaled to 5×5

A Quarter Five-spot Oil Reservoir Waterflood—6

Water saturation contours at 500 days



Fully upscaled to 5×5



Coarse 5×5

III. A Multiscale Mortar

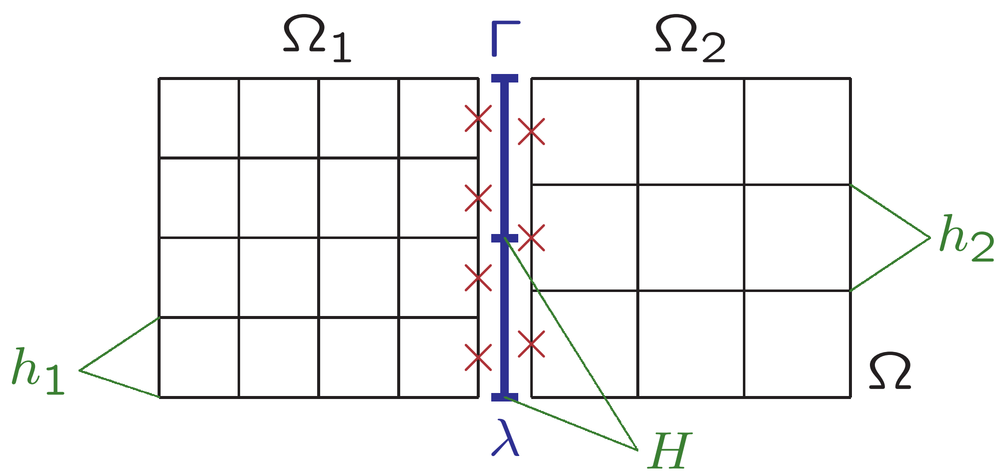
Mixed Finite Element Method

(with Pencheva, Wheeler, and Yotov)

Basic Idea of the Multiscale Mortar Mixed Method

1. **Localization.** Divide Ω into many small subdomains (or coarse elements of scale H), over which the original PDE is imposed.
2. **Fine-scale effects.** The subdomains are given Dirichlet boundary conditions $p = \lambda$ on Γ and solved on the fine scale h to define the local solution.
3. **Global coarse-grid problem.** The weakly defined flux mismatch (jump in $\mathbf{u} \cdot \nu$) on Γ is used to define a better λ on scale $H > h$, and we iterate the previous step until convergence is attained.
4. **Fine-grid construction.** We obtain a fully resolved and fully coupled approximate solution if λ is approximated in a higher order space.

By using a higher order mortar approximation, we compensate for the coarseness of the grid and maintain good (fine scale) overall accuracy.



Domain Decomposition Variational Form

Differential Equations.

$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \qquad \Gamma_i = \partial\Omega_i \setminus \Omega$$

$$\begin{cases} K^{-1}\mathbf{u} = -\nabla p & \text{in } \Omega_i & \text{(subdomain Darcy's law)} \\ \nabla \cdot \mathbf{u} = f & \text{in } \Omega_i & \text{(subdomain conservation)} \\ \mathbf{u}_i \cdot \nu_i + \mathbf{u}_j \cdot \nu_j = 0 & \text{on } \Gamma_{ij} & \text{(conservation on interface } \Gamma\text{)} \\ p|_{\Omega_i} = p|_{\Omega_j} & \text{on } \Gamma_{ij} & \text{(continuity of } p \text{ on } \Gamma\text{)} \\ p = 0 & \text{on } \partial\Omega & \text{(BC for simplicity)} \end{cases}$$

Variational form. Find $\mathbf{u} \in H(\text{div};\Omega_i)$, $p \in L^2(\Omega_i)$, $\lambda = p \in H^{1/2}(\Gamma_{ij})$:

$$\begin{cases} (K^{-1}\mathbf{u}, \mathbf{v})_{\Omega_i} = (p, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} & \forall\, \mathbf{v} \in H(\text{div};\Omega_i) \\ (\nabla \cdot \mathbf{u}, w)_{\Omega_i} = (f, w)_{\Omega_i} & \forall\, w \in L^2(\Omega_i) \\ \sum_i \langle \mathbf{u} \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0 & \forall \mu \in H^{1/2}(\Gamma_{ij}) \end{cases}$$

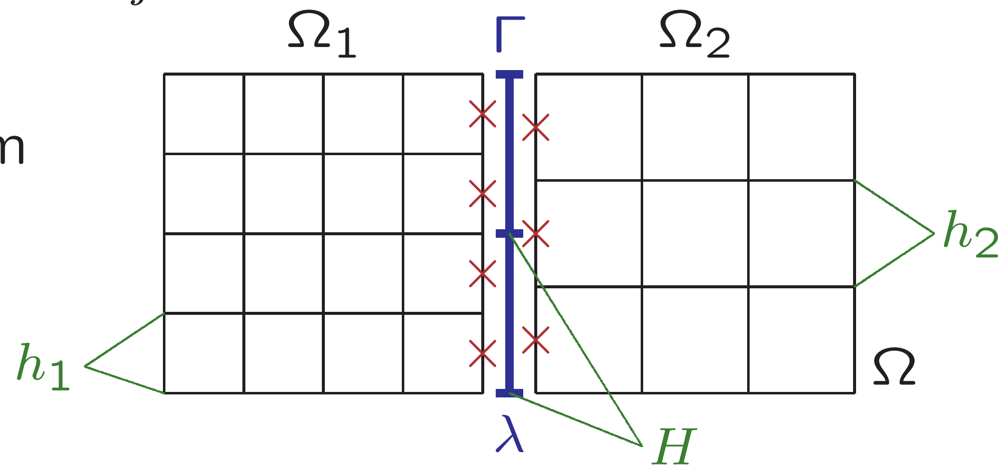
Remark. The last equation enforces continuity of flux on $\Gamma = \bigcup_i \Gamma_i$.

Finite element spaces.

- **Subdomain.** $\mathbf{V}_{h,i} \times W_{h,i}$ is the usual mixed space with polynomials of degree $k-1$ on mesh of spacing $h > 0$ on Ω_i.
- **Mortar.** $M_{H,ij}$ is continuous or discontinuous polynomials of degree $m-1$ on mesh of spacing $H > h$ on Γ_{ij}.

Key idea. On the interface

- Use only a few degrees of freedom (manage the linear algebra).
- Use higher order approximation (maintain accuracy).



Mortar method. Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, $\lambda_H \in M_H$ such that

$$\begin{cases} (K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda_H, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} & \forall\, \mathbf{v} \in \mathbf{V}_{h,i} \\ (\nabla\cdot\mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i} & \forall\, w \in W_{h,i} \\ \sum_i \langle \mathbf{u}_h\cdot\nu_i, \mu\rangle_{\Gamma_i} = 0 & \forall \mu \in M_H \end{cases}$$

Remark. The last equation enforces weak continuity of flux on Γ.

Implementation and Multiscale Finite Elements

Define the bilinear and linear forms on M_H by

$$d_H(\lambda,\mu) = \sum_i d_{H,i}(\lambda,\mu) = -\sum_i \langle \mathbf{u}_h^*(\lambda)\cdot\nu_i, \mu\rangle_{\Gamma_i}$$

$$g_H(\mu) = \sum_i g_{H,i}(\mu) = \sum_i \langle \bar{\mathbf{u}}_h\cdot\nu_i, \mu\rangle_{\Gamma_i}$$

where $(\mathbf{u}_h^*(\lambda), p_h^*(\lambda)) \in \mathbf{V}_h \times W_h$ solves (λ given, $f = 0$)

$$\begin{cases} (K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{v})_{\Omega_i} = (p_h^*(\lambda), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} & \forall\,\mathbf{v}\in\mathbf{V}_{h,i} \\ (\nabla\cdot\mathbf{u}_h^*(\lambda), w)_{\Omega_i} = 0 & \forall\, w\in W_{h,i} \end{cases}$$

and $(\bar{\mathbf{u}}_h, \bar{p}_h) \in \mathbf{V}_h \times W_h$ solves ($\lambda = 0$, f given)

$$\begin{cases} (K^{-1}\bar{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} = (\bar{p}_h, \nabla\cdot\mathbf{v})_{\Omega_i} & \forall\,\mathbf{v}\in\mathbf{V}_{h,i} \\ (\nabla\cdot\bar{\mathbf{u}}_h, w)_{\Omega_i} = (f, w)_{\Omega_i} & \forall\, w\in W_{h,i} \end{cases}$$

Theorem.

$$d_H(\lambda_H,\mu) = g_H(\mu) \quad \forall\mu\in M_H$$

if, and only if,

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_H) + \bar{\mathbf{u}}_h \qquad \text{and} \qquad p_h = p_h^*(\lambda_H) + \bar{p}_h$$

Domain Decomposition Iteration

(Glowinski & Wheeler, 1988; A., Cowsar, Wheeler & Yotov, 2000)

Interface problem. Find $\lambda_H \in M_H$ such that

$$d_H(\lambda_H, \mu) = g_H(\mu) \quad \forall \mu \in M_H$$

Theorem. The interface bilinear form $d_H(\cdot,\cdot)$ is symmetric and positive definite on M_H.

Thus, our problem reduces to a symmetric and positive definite linear system, and it can be solved by conjugate gradient iteration (for example). The computations involve:

- Once solving for $(\bar{\mathbf{u}}_h, \bar{p}_h)$ to get $g_H(\mu)$.
- Many times solving for $\left(\mathbf{u}_h^*(\lambda_H^i), p_h^*(\lambda_H^i)\right)$ to get $d_H(\lambda_H^i, \mu)$.

This seems like a natural way to obtain the solution.

Mortar Degrees of Freedom

Let $\{\mu_\ell\}$ be a basis for $M_H = \text{span}\{\mu_\ell\}$. Define

$$\mathbf{v}_\ell = \mathbf{u}_h^*(\mu_\ell) \qquad \text{and} \qquad w_\ell = p_h^*(\mu_\ell)$$

Then

$$\lambda_H = \sum_\ell \lambda_\ell \mu_\ell \quad \text{and} \quad \mathbf{u}_h = \sum_\ell \lambda_\ell \mathbf{v}_\ell + \bar{\mathbf{u}}_h \quad \text{and} \quad p_h = \sum_\ell \lambda_\ell w_\ell + \bar{p}_h$$

Find $\{\lambda_\ell\}$ such that

$$\sum_\ell \lambda_\ell \, d_H(\mu_\ell, \mu_k) = g_H(\mu_k) \quad \forall k$$

is equivalent to

$$(K\nabla p, \nabla q) = \sum_\ell \lambda_\ell (K^{-1}\mathbf{v}_\ell, \mathbf{v}_k) = (f, w_k)$$

This is another natural way to solve the problem!

Let

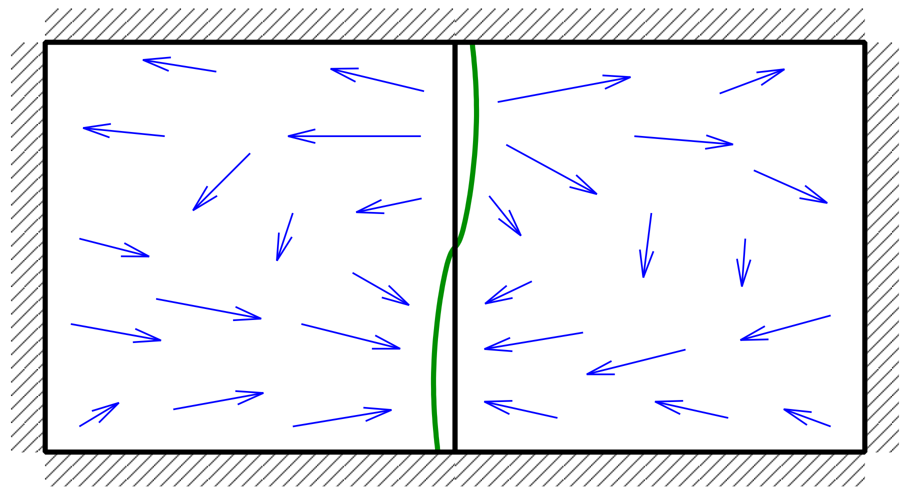
$$N_{h,H} = \text{span}\left\{ \begin{pmatrix} \mathbf{v}_\ell \\ w_\ell \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} \mathbf{u}_h^*(\mu_\ell) \\ p_h^*(\mu_\ell) \end{pmatrix} \right\} \subset \begin{pmatrix} \mathbf{V}_h \\ W_h \end{pmatrix}$$

Multiscale finite element formulation. Find $\begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} \in N_{h,H} + \begin{pmatrix} \bar{\mathbf{u}}_h \\ \bar{p}_h \end{pmatrix}$ so that

$$(K^{-1}\mathbf{u}_h, \mathbf{v}) = (f, w) \quad \forall \begin{pmatrix} \mathbf{v} \\ w \end{pmatrix} \in N_{h,H}$$

Remarks. This is an unusual multiscale finite element method!

- We couple pressures and velocities.
- We allow flow on all edges.
- We add a constant term to the solution, as is typical of variational multiscale methods.
- Our multiscale finite elements are locally defined over the subdomains (i.e., the course elements).



Analysis of Errors

A-Priori Error Estimates

Theorem. There exists C, independent of h and H, such that

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C\Big\{\|\mathbf{u}\|_k h^k + \|p\|_{m+1/2} H^{m-1/2} + \|\mathbf{u}\|_{k+1/2} h^k H^{1/2}\Big\}$$

$$= \mathcal{O}\bigg(\Big(\frac{h}{\epsilon}\Big)^k + \frac{1}{\epsilon}\Big(\frac{H}{\epsilon}\Big)^{m-1/2} + \Big(\frac{h}{\epsilon}\Big)^k\Big(\frac{H}{\epsilon}\Big)^{1/2}\bigg)$$

$$\|p-p_h\|_0 \le C\Big\{\|p\|_k h^k + \|p\|_{m+1/2} H^{m+1/2} + (\|f\|_k + \|\mathbf{u}\|_k) h^k H + \|\mathbf{u}\|_{k+1/2} h^k H^{3/2}\Big\}$$

$$= \mathcal{O}\bigg(\Big(\frac{h}{\epsilon}\Big)^k + \Big(\frac{H}{\epsilon}\Big)^{m+1/2} + h\Big(\frac{h}{\epsilon}\Big)^{k-1}\Big(\frac{H}{\epsilon}\Big)^{3/2}\bigg)$$

Remark. We can also obtain superconvergence results.

Problem of Scale and Adaptivity. We turn to an a-posteriori error analysis and an iterative grid refinement process to resolve the coupling dynamics.

Explicit Residual-Based Estimators: Upper Bounds

For all $E \in \mathcal{T}_h^\Omega$ and $\tau \in \mathcal{T}_H^\Gamma$,

$$\omega_E^2 = \|K^{-1}\mathbf{u}_h + \nabla p_h\|_E^2 h_E^2 + \|f - \nabla\cdot\mathbf{u}_h\|_E^2 h_E^2 \quad \text{(Residuals)}$$

$$+ \|\lambda_H - p_h\|_{\partial E\cap\Gamma}^2 h_E \quad \text{(Pressure mismatch)}$$

$$\omega_\tau^2 = \sum_{E\in E_\tau} \|[\mathbf{u}_h\cdot\nu]\|_{\partial E\cap\tau}^2 H_\tau^3 \quad \text{(Flux mismatch)}$$

Theorem. There exists a constant C, independent of h and H, such that

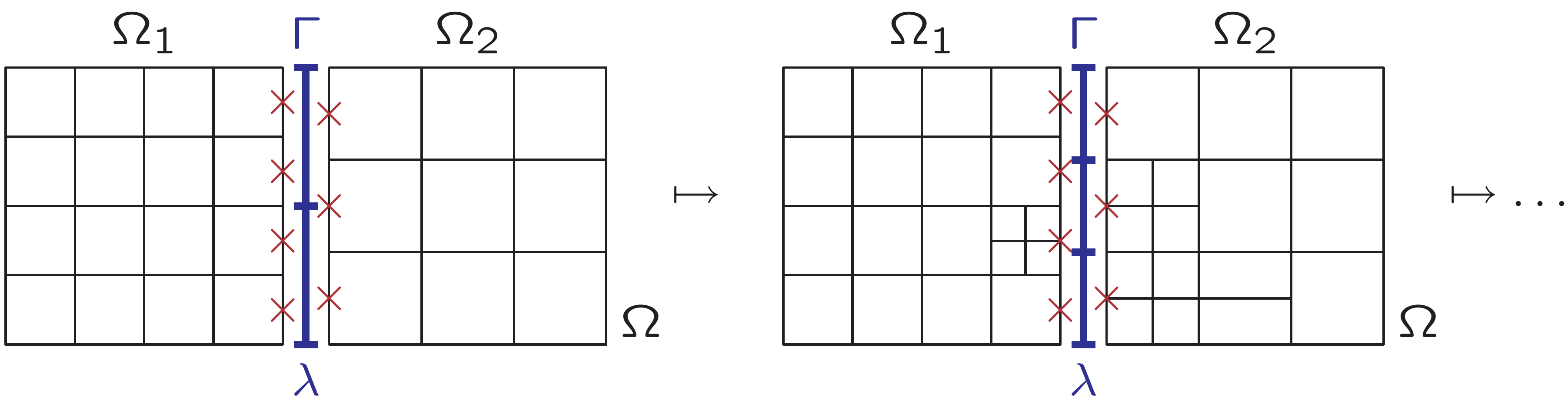
$$\|p - p_h\|_0 \le C\left\{\sum_{E\in\mathcal{T}_h^\Omega}\omega_E^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma}\omega_\tau^2\right\}^{1/2}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C\left\{\sum_{E\in\mathcal{T}_h^\Omega} h_E^{-2}\omega_E^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma} H_\tau^{-2}\omega_\tau^2\right\}^{1/2}$$

Saturation Assumptions. We need sufficient resolution that the coarse approximation contains some "reasonable" information about the solution, so that we can detect inadequate resolution. These saturation assumptions are justified by the a-priori error theorem.

This analysis is *not* a standard multiscale analysis.

- A-posteriori error indicators are computed from the input data and the computed solution.
- The error indicators drive adaptive mesh refinement (AMR).
- Through AMR iteration, the numerical solution is obtained on appropriate subdomain and mortar grids.



Remark: We detect the multiscale nature of the solution through this a-posteriori analysis of AMR intermediate approximation results.

Some Numerical Results

A 2-D smooth problem with known analytic solution

$$p(x,y) = x^3y^4 + x^2 + \sin(xy)\cos(y)$$

and full tensor coefficient

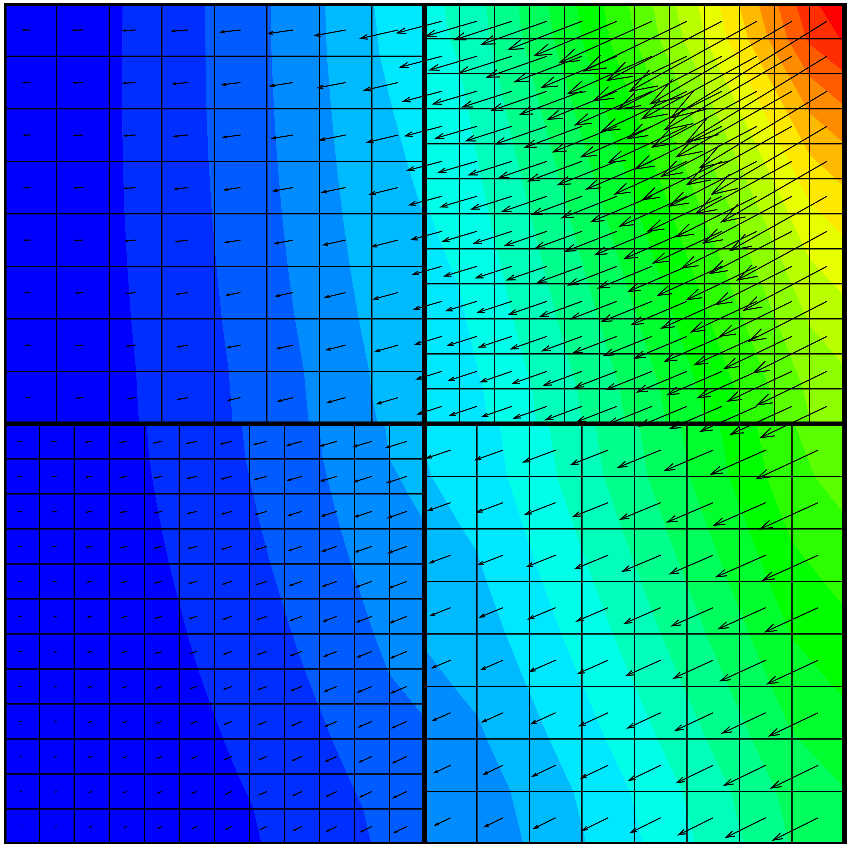
$$K = \begin{pmatrix} (x+1)^2 + y^2 & \sin(xy) \\ \sin(xy) & (x+1)^2 \end{pmatrix}.$$

Remarks.

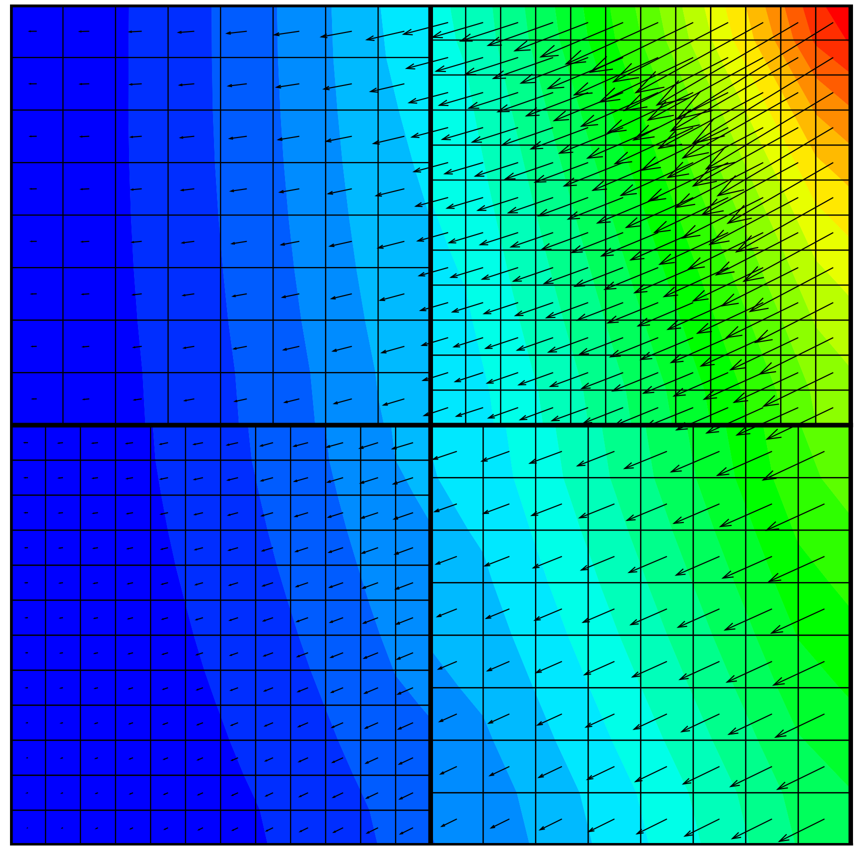
- Use lowest order Raviart-Thomas spaces (RT0).
- Solve using conjugate gradients and a balancing preconditioner.
- Use the scaling $H = h^{1/2}$ for $m = 3$ and $H = 2h$ for $m = 2$, which is optimal for superconvergent velocities.

Ex. 1—A Smooth Case with a Full Tensor—2

Computed pressure and velocity on nonmatching grids.



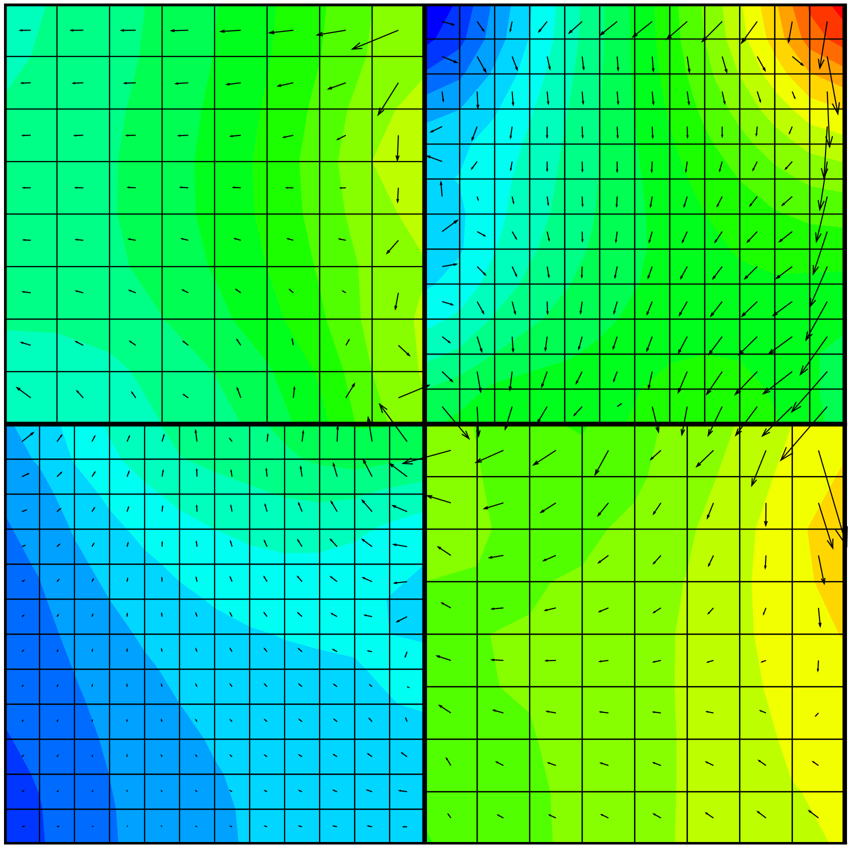
Discontinuous quadratic mortars



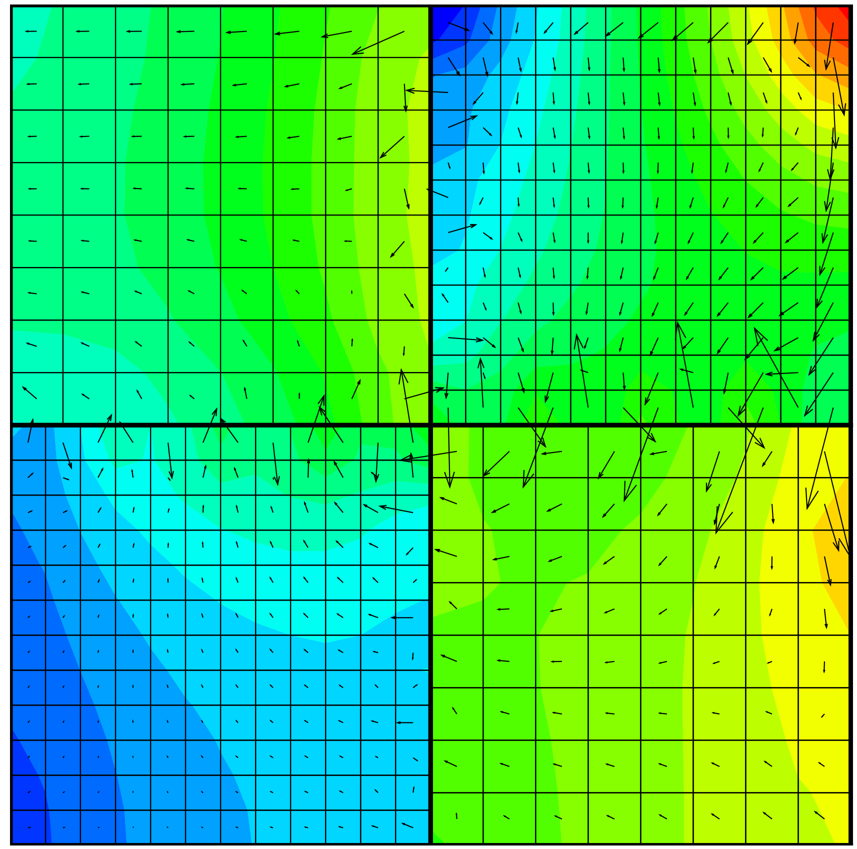
Discontinuous linear mortars

Ex. 1—A Smooth Case with a Full Tensor—3

Error in pressure and velocity on nonmatching grids.



Discontinuous quadratic mortars



Discontinuous linear mortars

Conclusion. Quadratic mortars do a better job near the interface Γ.

Discontinuous quadratic mortars and nonmatching grids.

$1/h$	Iter	Cond	$\|\|p-p_h\|\|$	$\|\|\mathbf{u}-\mathbf{u}_h\|\|$	$\|\|\|p-p_h\|\|\|$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$
4	8	18.8	2.64E-1	2.03E-1	4.62E-2	2.13E-2
16	7	2.5	6.37E-2	4.86E-2	2.83E-3	1.82E-3
64	7	2.3	1.59E-2	1.21E-2	1.75E-4	1.59E-4
256	8	3.0	3.98E-3	3.03E-3	1.09E-5	1.68E-5
Rate			1.01	1.01	2.01	1.72
Theor			1.00	1.00	1.50	1.25

Discontinuous linear mortars and nonmatching grids.

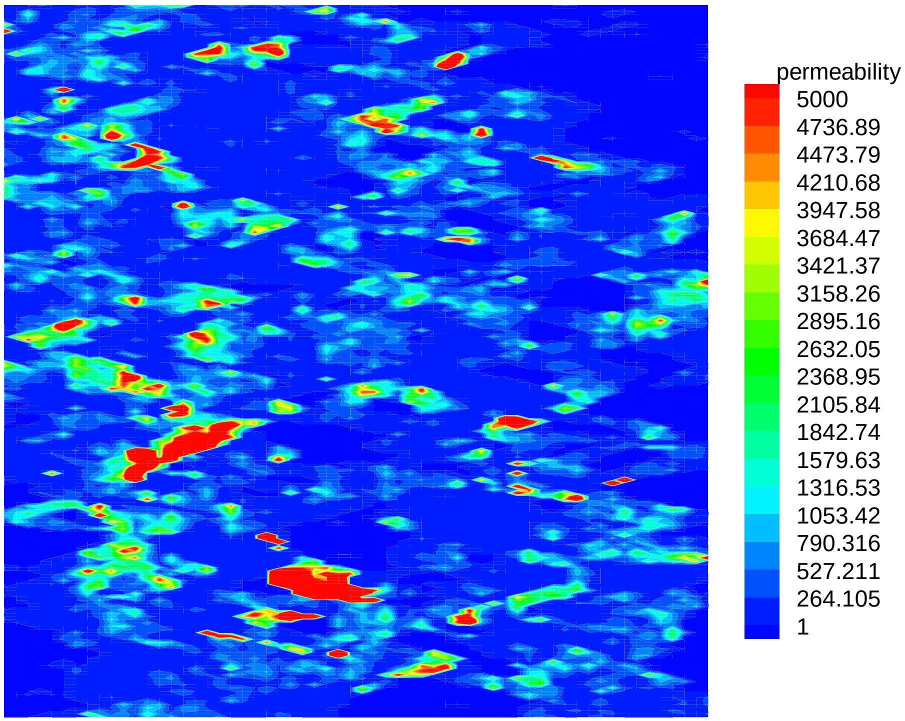
$1/h$	Iter	Cond	$\|\|p-p_h\|\|$	$\|\|\mathbf{u}-\mathbf{u}_h\|\|$	$\|\|\|p-p_h\|\|\|$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$
4	4	1.31	2.63E-1	2.04E-1	4.54E-2	2.35E-2
16	7	2.12	6.37E-2	4.86E-2	2.82E-3	2.30E-3
64	8	3.27	1.59E-2	1.21E-2	1.75E-4	2.38E-4
256	8	5.02	3.98E-3	3.03E-3	1.09E-5	2.74E-5
Rate			1.01	1.01	2.01	1.63
Theor			1.00	1.00	2.00	1.50

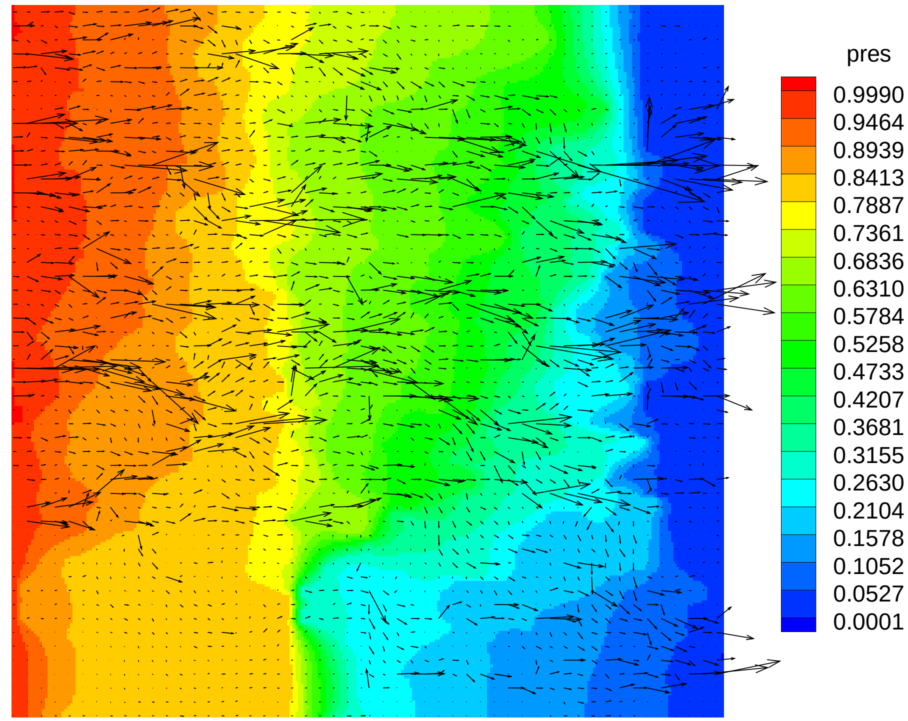
Conclusions.

- The solution procedure is efficient (# iterations $\sim$ constant).
- Continuous and discontinuous mortars give similar errors.
- Matching or nonmatching of the subdomain grids is not important.

Ex. 2—A Highly Heterogeneous Case

A heterogeneous permeability from the 2001 SPE Comparative Solution Project 10. It varies more than five orders of magnitude.



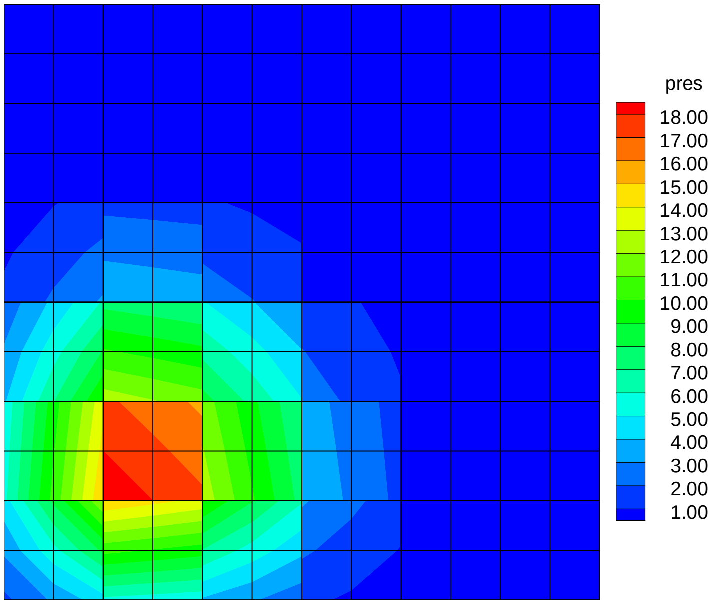


$1/h$	Iter	Cond
4	4	8.6
16	17	48.7
64	22	45.1
256	26	31.2

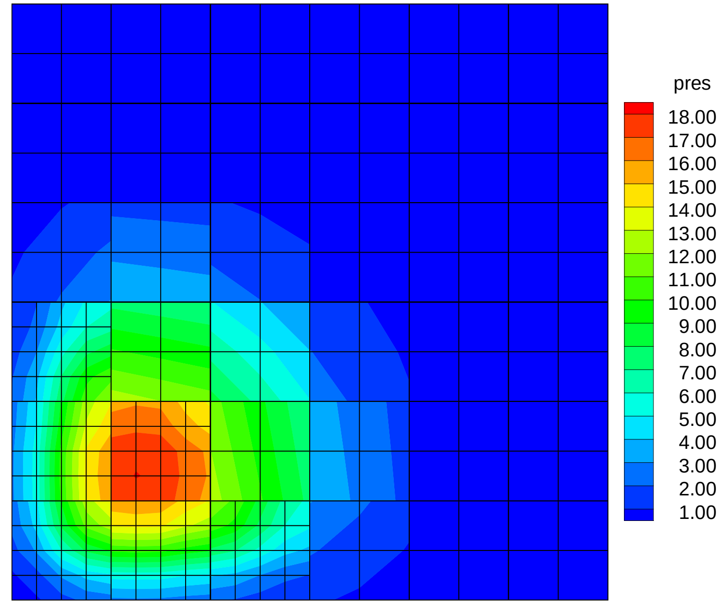
Pressure and velocity.
Discontinuous quadratic mortars and matching grids.

Ex. 3—A Boundary Layer Case

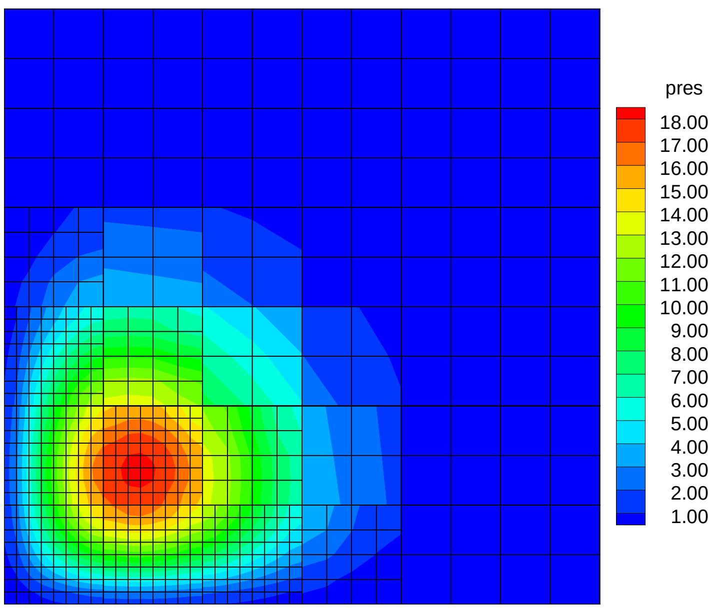
Pressures with single discontinuous quadratic mortar on each interface.



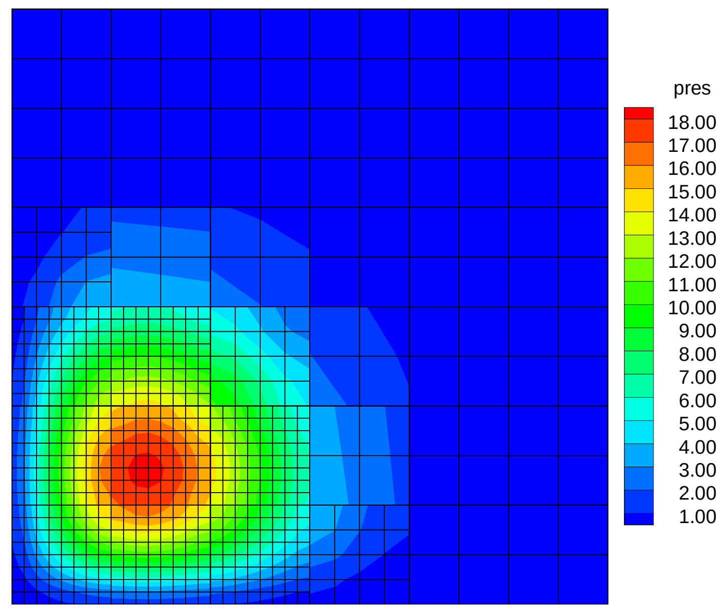
Refinement level 1



Refinement level 2



Refinement level 3



Refinement level 4

Summary and Conclusions

Summary and Conclusions

1. Multiscale Numerical Method handle scales through:
   - **Localization** into small coarse elements or subdomains;
   - **Fine-scale effects** through solution of the local subproblems;
   - **Global coarse-grid problem** using the fine-scale information;
   - **Fine-grid construction** of the approximate solution.

2. Multiscale finite elements resolve fine scales on coarse grids, and they converge with respect to the scale of heterogeneity.

3. Multiscale mortar methods reduces to a linear system for the interface mortar degrees of freedom and is solved either using
   - domain decomposition iteration;
   - or multiscale finite elements that couple pressures and velocities.

   A nonstandard multiscale analysis shows that a-posteriori error indicators can be used to iteratively adapt the mesh to the solution.

4. Numerical results show that we can resolve the main components of the flow for very large problems on very coarse grids, even though we under-resolve the fine scales themselves.

SIAM Conference on Mathematical & Computational Issues in the Geosciences

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Consider

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- submitting a paper
- organizing a minisymposium