Multiscale finite element methods using limited global information

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Introduction

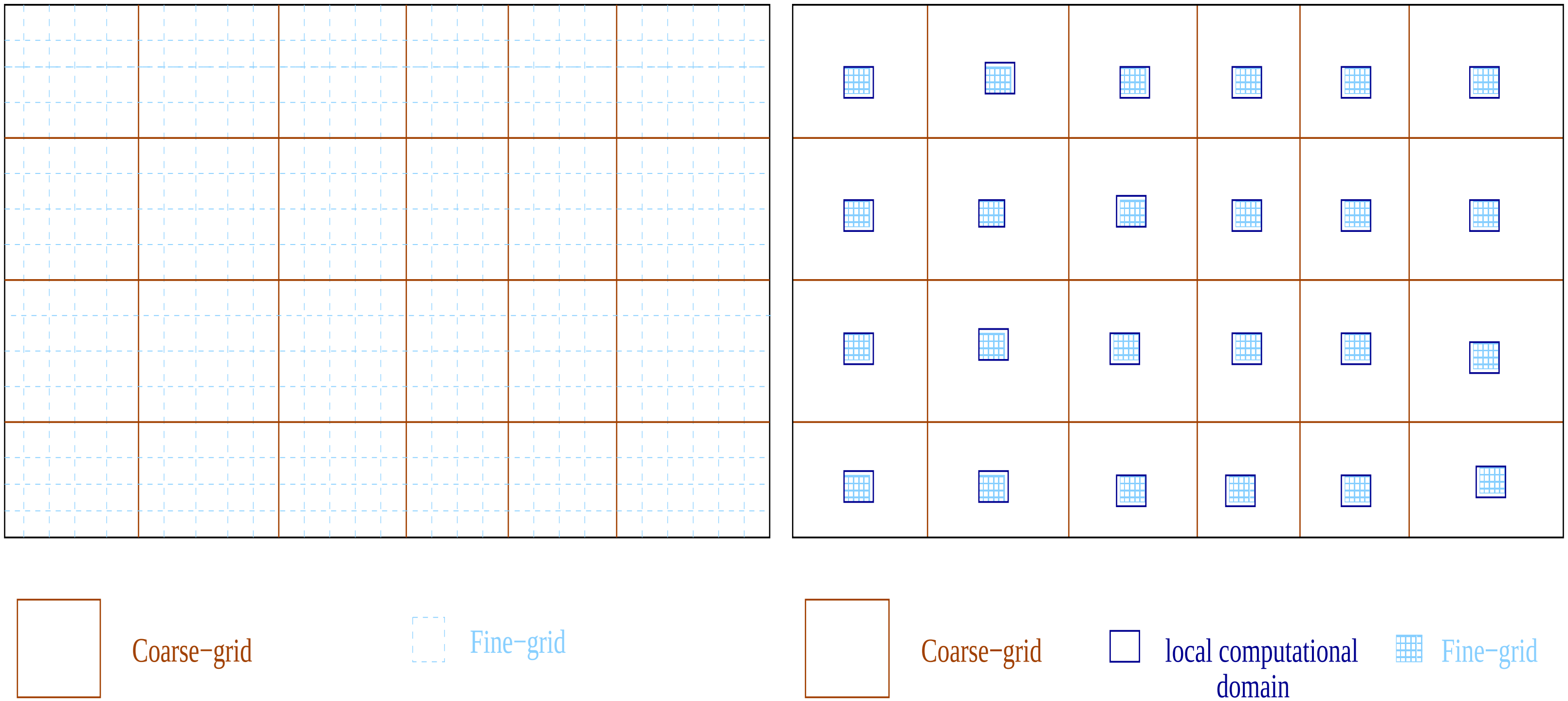
- Because of a wide range of scales, direct numerical simulations are not affordable for subsurface simulations, model calibration, uncertainty quantification, reservoir optmization.
- Typically some type of coarsening is needed. Two type of approaches are commonly used: upscaling and multiscale.
- Some of the main challenges in coarsening subsurface processes are modeling the effects of non-separable scales and uncertainties.
- In this talk, we discuss approaches which capture the effects of spatial scales and uncertainties.

Coarse and fine

Consider

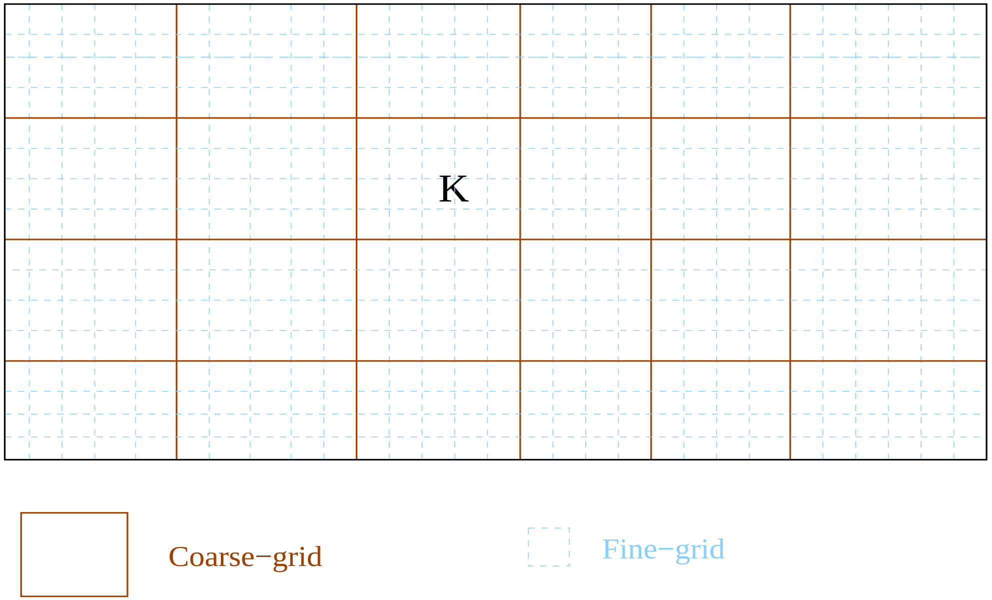
$$Lp = f,$$

where $Lp = -div(k(x)\nabla p)$. Our goal is to solve this equation on a coarse grid.



Schematic description of coarse grid

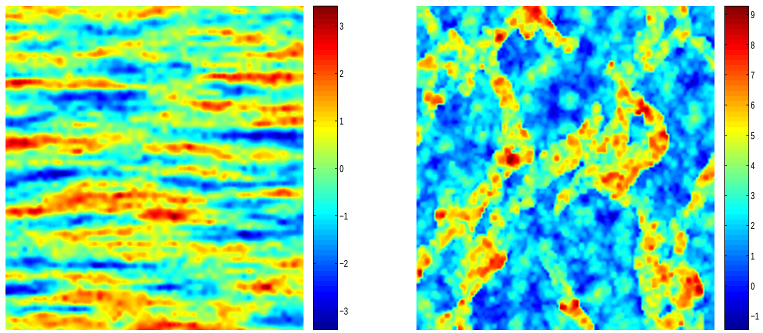
Upscaling



- Consider $div(k\nabla p) = f$.
- For each coarse grid block, $div(k\nabla \phi_i) = 0$ is solved in K with $\phi_i = x_i$ on ∂K.
- $k^*_{\cdot,i} = \frac{1}{|K|}\int_K k\nabla\phi_i dx$
- The coarse-scale equation $div(k^*\nabla p^*) = f$ is solved to find the upscaled solution.

Requirements/Challenges

- Upscaling (multiscale) computes local effective parameters (basis functions) that can be used repeatedly for different source/boundary conditions and the local changes only affect a few parameters (basis functions).
- Accuracy and Robustness
- Valid for different types of subsurface heterogeneity



- Some advantages: (1) parallel multiscale basis function construction (which can be very cheap if there is scale separation); (2) re-use of basis functions; (3) adaptive downscaling

Outline

- Multiscale finite element methods (MsFEMs). Subgrid capturing errors and possible improvements.
- Extension of MsFEMs to nonlinear problems
- MsFEMs using limited global information for problems with non-separable scales.
- Ensemble level MsFEMs
- Applications to two- and three-phase flow and transport.
- Coarse gridding.

Multiscale finite element methods

Hou and Wu (1997) (cf. Babuska and Osborn, 1984).

- Multiscale finite element methods have 2 ingredients: (1) multiscale basis functions; (2) a choice of global coupling of these basis functions.
- Basis functions attempt to capture the small scale information of the multiscale process under the consideration and allow us to perform downscaling.
- Basis functions are constructed by solving the leading order homogeneous equation in an element K (coarse grid or RVE)

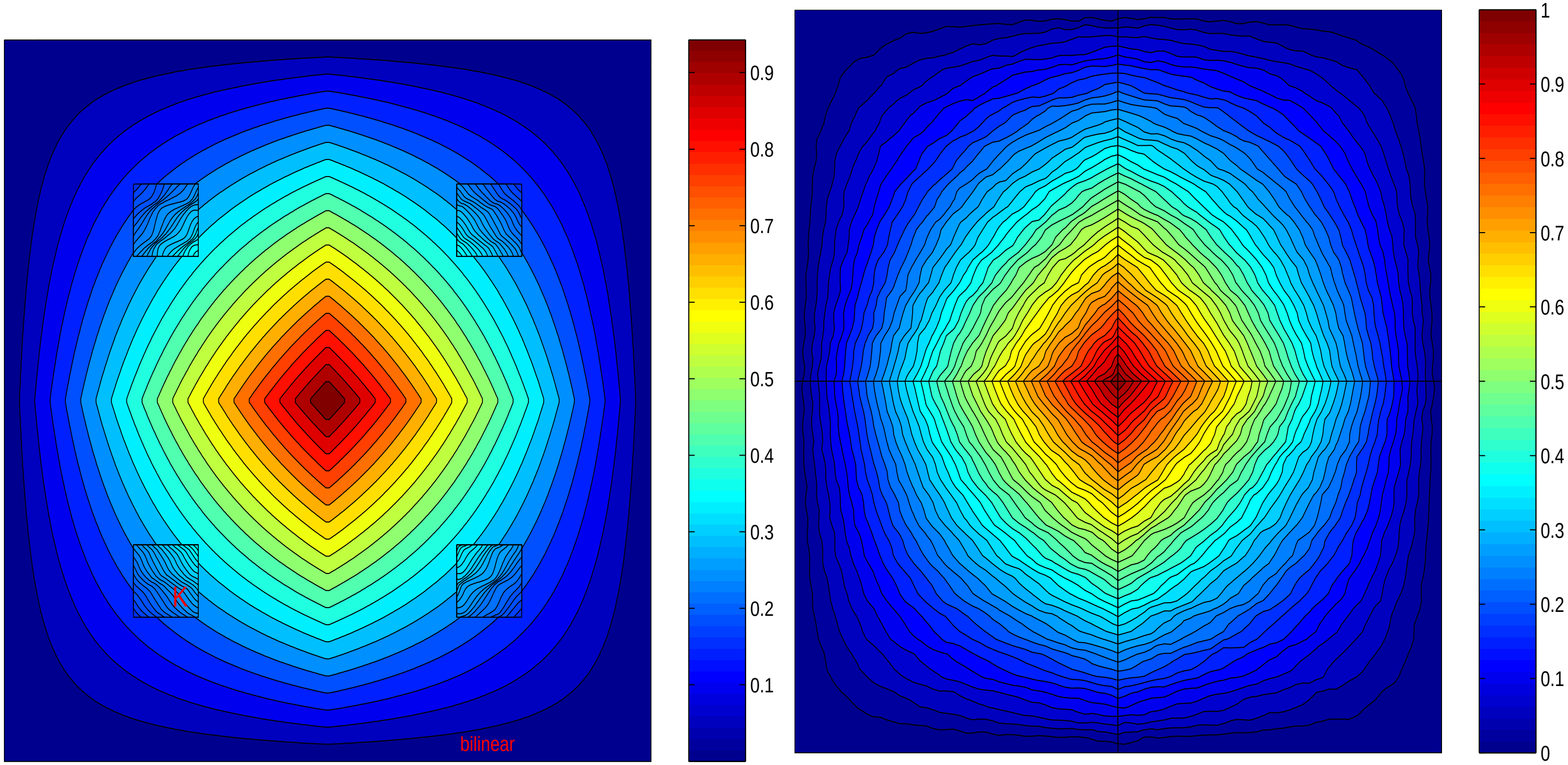
$$L(\phi_i) = 0 \quad \text{in} \quad K,$$

  K is a coarse grid block.
- Boundary conditions for the basis functions are very important.
- Basis functions are coupled via a global variational formulation. $p \approx \sum_i p_i \phi_i$, where

$$\langle L(\sum_i p_i \phi_i), \tilde{\phi}_j \rangle = \langle f, \tilde{\phi}_j \rangle,$$

  e.g., $\tilde{\phi}_i$ are standard piece-wise linear basis functions.

Basis functions



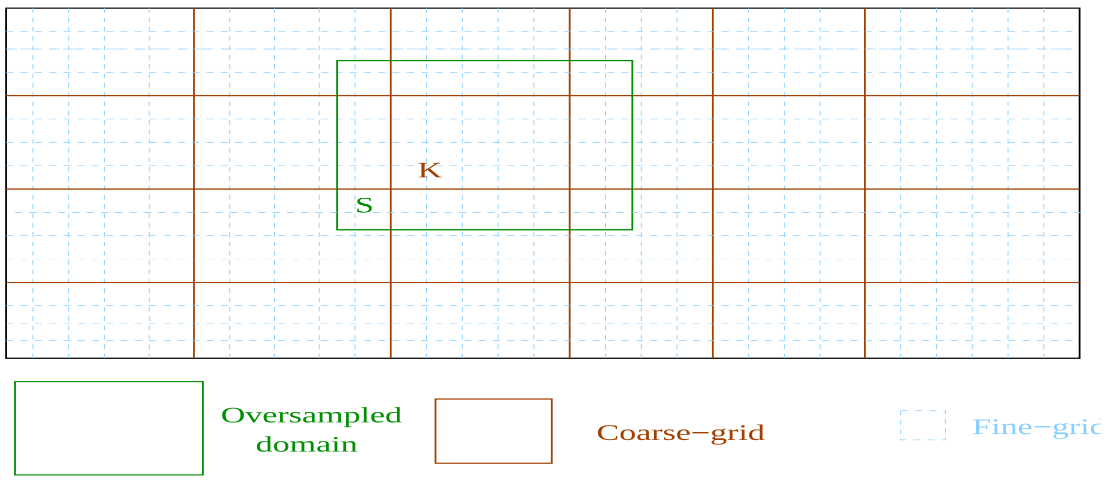
Example of basis functions. Left: basis function with K being RVE. Right: basis function with K being a coarse element

Relevant methods

- Classical upscaling or numerical homogenization.
- Subgrid modeling (by T. Arbogast, I. Babuska, T. Hughes, ...)
- Multiscale finite volume methods (P. Jenny, H. Tchelepi, S.H. Lee and others).
- Subgrid stabilization (by F. Brezzi, L Franco, T. Hughes, ...).
- Heterogeneous multiscale methods (E, Engquist,...)
- Numerical homogenization using two-scale convergence (C. Schwab, V.H. Hoang,...)
- Numerical homogenization (A. Bourgeat, A. Madureira, M. Sarkis, Cao, ...)
- Mortar upscaling (T. Arbogast, M. Peszynska, M. Wheeler, I. Yotov,...)
- Variational coarsening (D. Moulton and S. P. MacLachlan)
- ...

Capturing subgrid effects. Localization errors

- What is the effect of local (polynomial, reduced) boundary conditions on the convergence of MsFEM?
- The effects of local boundary conditions are studied for e.g., periodic problems, such as $k(x, x/\epsilon)$.
- It was shown (Hou, Wu, Cai, Math. Comp. 1999) that MsFEMs suffer from resonance errors ($(\frac{\epsilon}{h})^{\beta}$) due to the fact that the boundary conditions of basis functions do not match oscillations which are present in the global solution.
- A remedy is to use larger domains in constructing multiscale basis functions.



Extension to nonlinear problems

- In a number of papers with my collaborators, we have extended MsFEMs to nonlinear problems and analyzed these methods. In this extension, multiscale basis functions are replaced by multiscale maps.
- Consider $Lp = f$, where L is a nonlinear operator (e.g., $L(p) = p_t - div a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$). $L : X \to Y$.
- For each element $v_h \in W_h$ (where W_h is a usual FE space on the coarse space), $v_{r,h} = E^{MsFEM} v_h$ is defined as

$$L^{map} v_{r,h} = 0 \text{ in } K,$$

  where L^{map}, in general, is different from L.
- Find $p_h \in W_h$ such that

$$\langle L^{global} p_{r,h}, v_h \rangle = \langle f, v_h \rangle, \ \forall v_h \in W_h,$$

  where $\langle \cdot, \cdot \rangle$ denotes a duality between X and Y L^{global} is different from L, in general.
- These methods are analyzed for monotone and pseudomonotone operators.

Two-phase flow and transport

A protypical example is two-phase immiscibe flow (e.g., water and oil).

$$-div(\lambda(S)k(x)\nabla p) = q, \quad \frac{\partial}{\partial t}S + v \cdot \nabla f(S) = q_w,$$

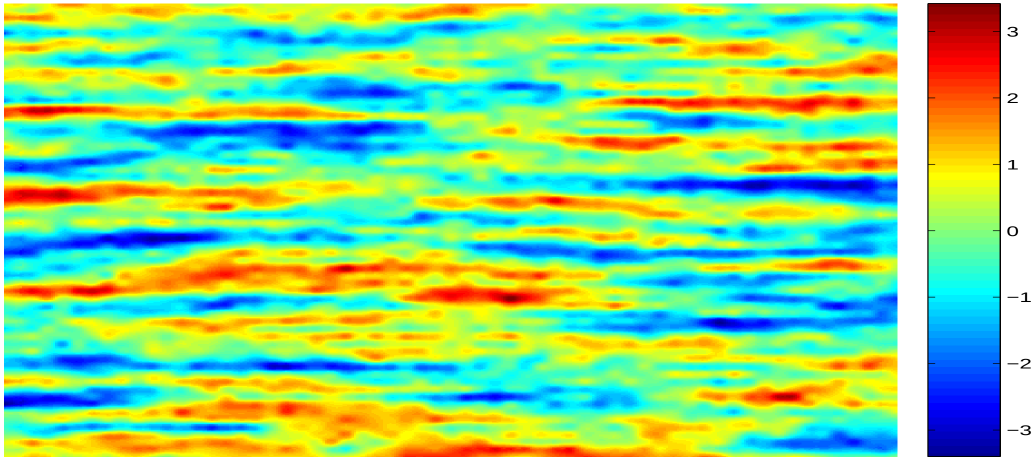
where $v = -\lambda(S)k(x)\nabla p$.

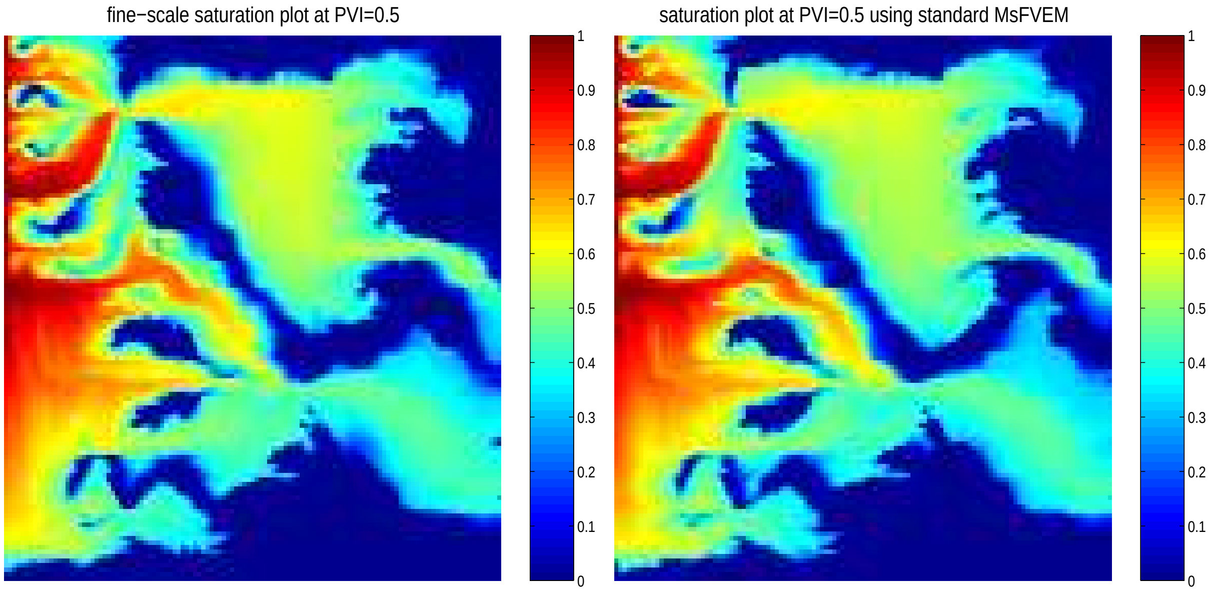
- Various upscaling methods are developed for multi-phase flow and transport, e.g., single-phase upscaling, multi-phase upscaling, ...
- We will coarsen the pressure equation and solve the saturation equation on the fine grid.
- Note that the pressure equation has the form

$$-div(\lambda(x,t)k(x)\nabla p) = q,$$

where $\lambda(x,t)$ is smooth and changes in time. We will compute the basis functions at time zero and use it throughout the simulations with changing them.

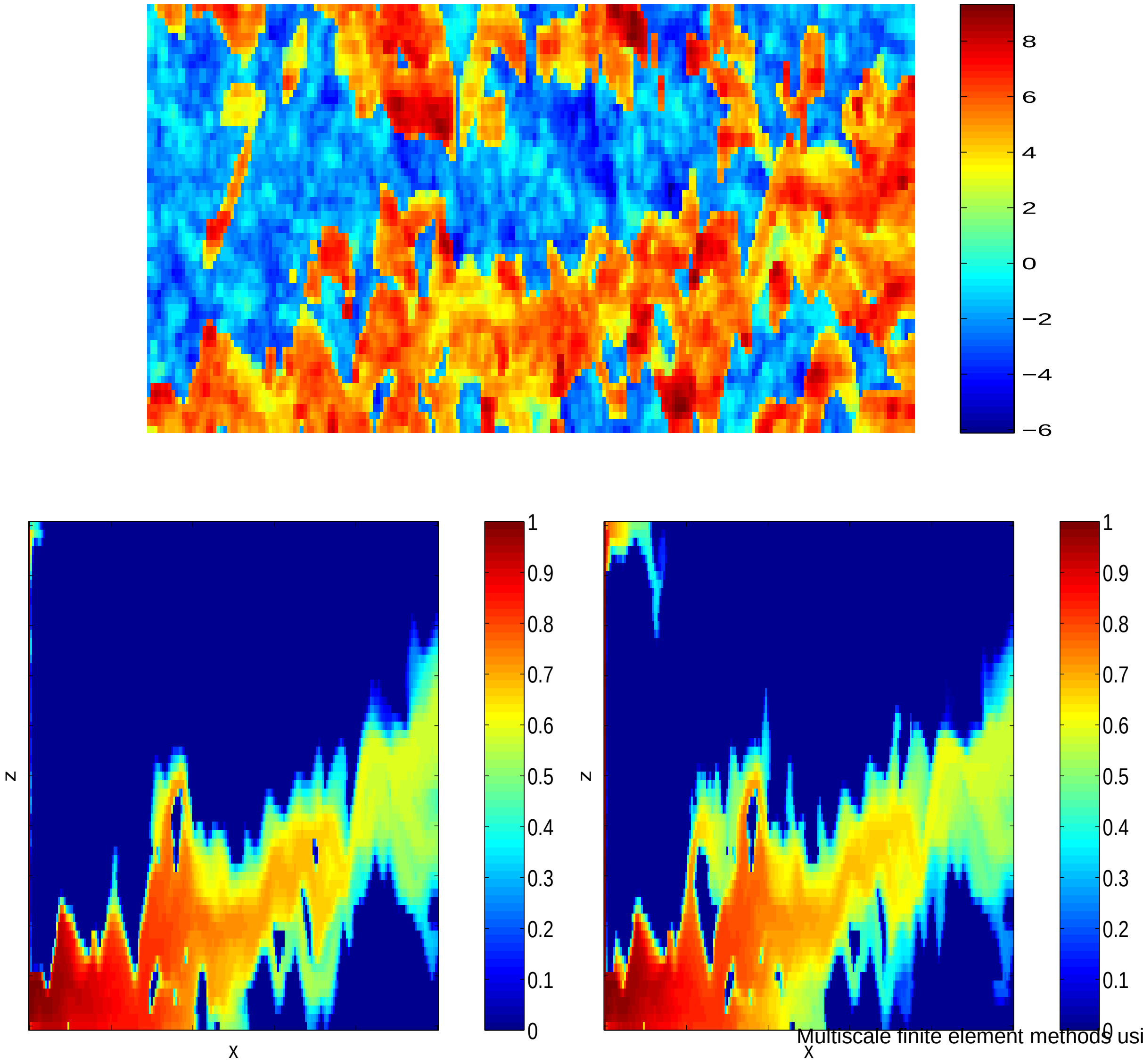
Two-point geostatistics





Channelized permeability fields

Benchmark tests: SPE 10 Comparative Project



The use of limited global information

- The numerical tests using strongly channelized permeability fields (such as SPE 10 Comparative) show that local basis functions can not accurately capture the long-range information. There is a need to incorporate non-local information.
- Construct basis functions to approximate

$$div(\lambda(x)k_r(x)\nabla p) = g(x),$$

for any $\lambda(x)$ and $g(x)$ that vary on the coarse grid (smooth function).

- Some type of global information is needed, e.g.,

$$div(k_r(x)\nabla p^0) = f(x),$$

$p^0(x) = p(x)$ on ∂Q.

- Previous work: L. Durlofsky et al., 1997,... Perform coarsening and coarse-scale parameters based on "single-phase" solutions. Later on: Y. Chen and Durlofsky, 2005; J. Aarnes, 2004; X.H. Wu et al. 2006;

The use of global information

- For subsurface applications, we need numerical methods which (1) are conservative (i.e., reconstructed fine-scale velocity field $v = -k\nabla p$ is a conservative field) (2) can handle unstructured fine and coarse grids.
- Assume that the important global information can be incorporated into some spatial fields, p_1, ..., p_N. More precisely, we assume

  $$p \approx G(p_1, ..., p_N)$$

  where G is a smooth function.
- Coarse grid block is a connected union of fine grid blocks.

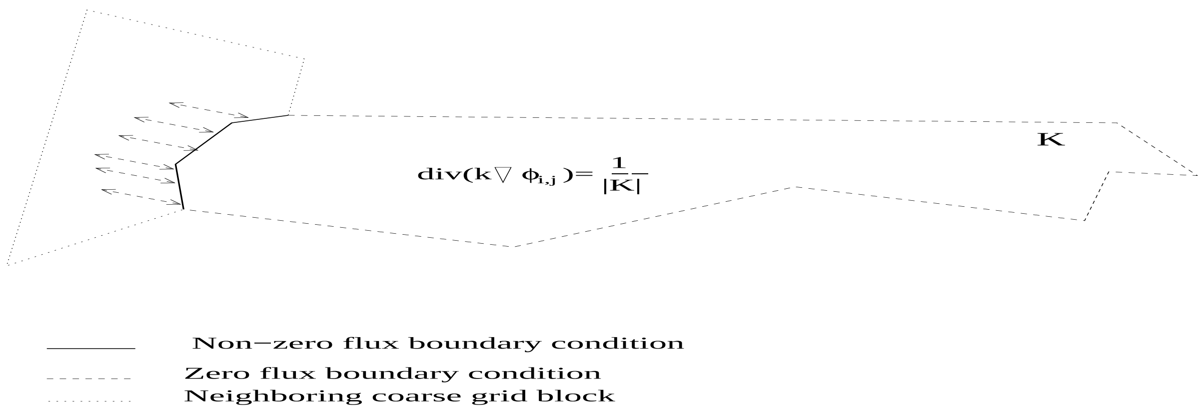
Basis construction

- Consider $v_i = k\nabla p_i$. Then, the velocity basis functions are given by $\Psi_{e_i} = k(x)\nabla\phi$, where

$$div(k(x)\nabla\phi) = \frac{1}{|K|} \quad \text{in} \quad K$$

  and $k(x)\nabla\phi \cdot n = \frac{v_i \cdot n}{\int_{e_l} v_i \cdot n ds}$ on e_i and 0 otherwise.

- Pressure is discretized by piecewise constants.
- The multiscale basis functions are coupled via the mixed finite element framework.
- Unstructured (corner-point) fine-grid is used. The requirement for a coarse grid is that it is a connected union of fine grid blocks. Mimetic FD is used for solving local problems.



Global fields

For a realization of the media, p_i can be

- (1) the solution of single-phase (it can be shown that for channelized media, $p^{tp} \approx G(p^{sp}, t)$, where $div(k \nabla p^{sp}) = 0$ (Efendiev et al., 2006).
- (2) the solutions of directional flows (Owhadi and Zhang, 2006). It is shown that $p = \hat{p}(u_1, u_2) \in W^{2,s}(Q)$, where $-div(k(x) \nabla u_i) = 0$, $u_i = x_i$
- (3) the solution in RVE (in ϵ-period) in the case of scale separation.

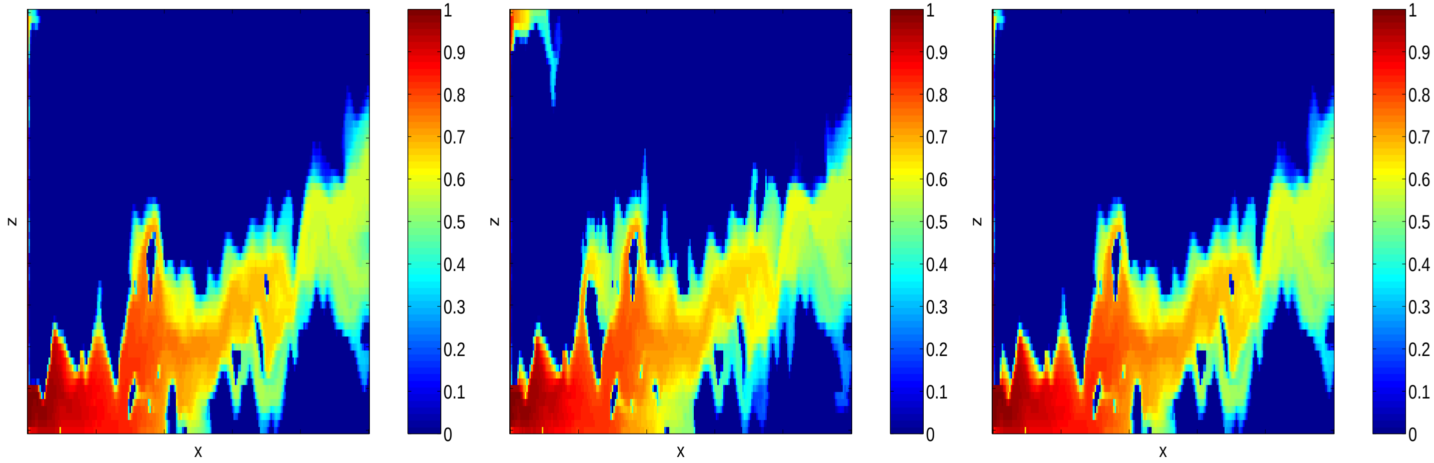
For stochastic media properties, p_i are representative fields representing the stochastic multiscale solutions across the ensemble.

- The convergence independent of resonance errors has been shown in Aarnes, Efendiev and Jiang, SIAM MMS 2008 for quasiuniform meshes as well as for highly anisotropic grids.
- No need for matching grids.

Approximate global fields

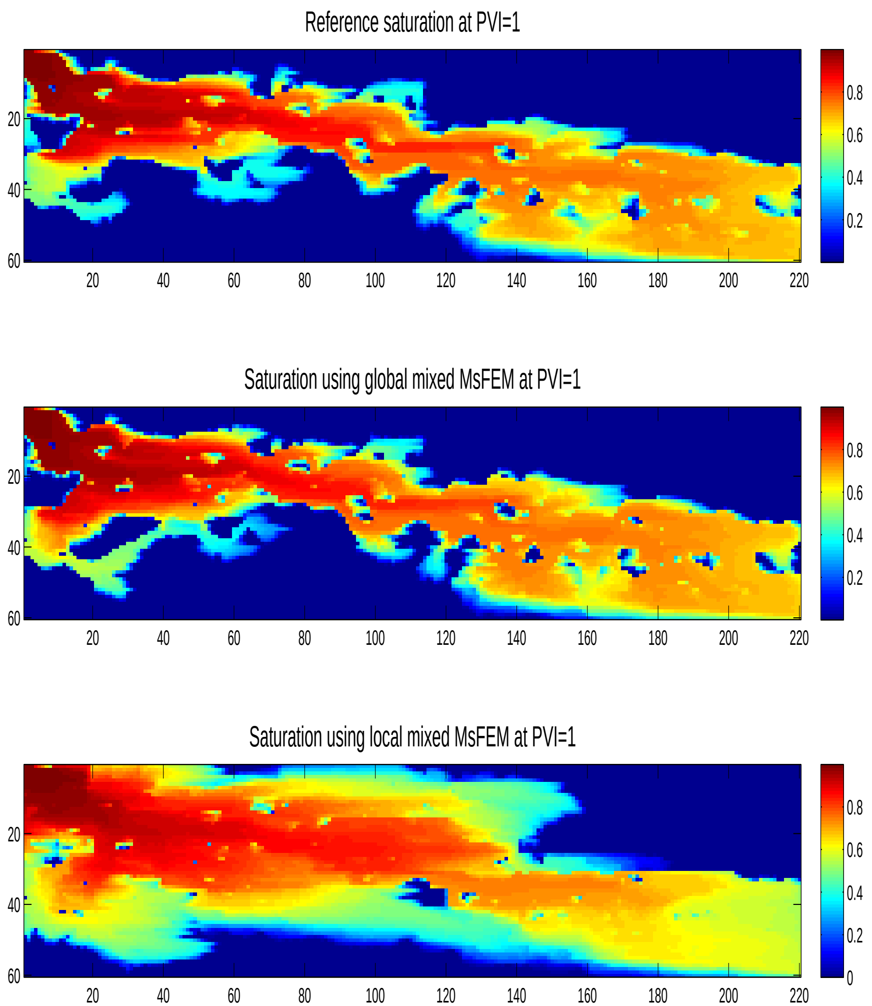
- “Global” fields represent some important features which “can not be” captured via local approximations.
- Approximate global fields can be used in the simulations.
- *Iterative MsFEM*. The global fields are obtained via the solutions of MsFEM with oversampling. This is iterated until convergence (Durlofsky et al., 2007, Hajibeygi et al., 2008)
- Global fields are computed via upscaling of fine-scale problems to a grid finer than the coarse grid. The objective is to preserve long-range information in permeability fields.
- For the analysis, $k_{\delta>,\epsilon}$ is used as the reference permeability and $k^*_{\delta>}$ for a coarse-scale permeability (Efendiev and Jiang, 2008).

Channelized reservoir



Comparison of saturation profile at PVI=0.5: (left) fine-scale model, (middle) standard MsFVEM, (right) modified MsFVEM

Numerical results

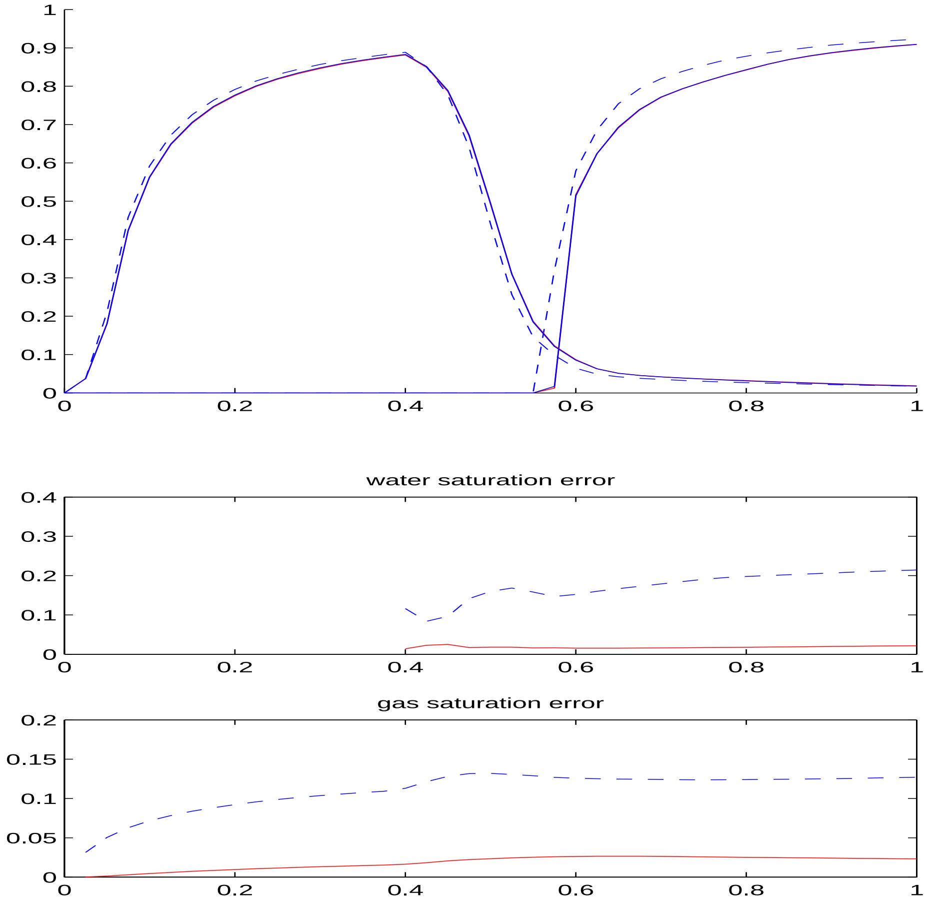


Numerical results

Relative Errors (layer=40, $\frac{\mu_o}{\mu_w} = 3$)

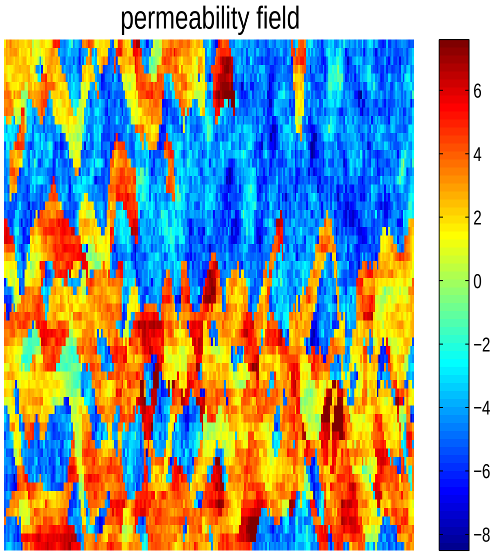
coarse grid	frac. flow error (global)	saturation error (global)	frac. flow error (local)	saturation error (local)
6×10	0.0144	0.0512	0.1172	0.2755
12×11	0.0093	0.0435	0.2057	0.3459
12×22	0.0039	0.0370	0.1867	0.3158

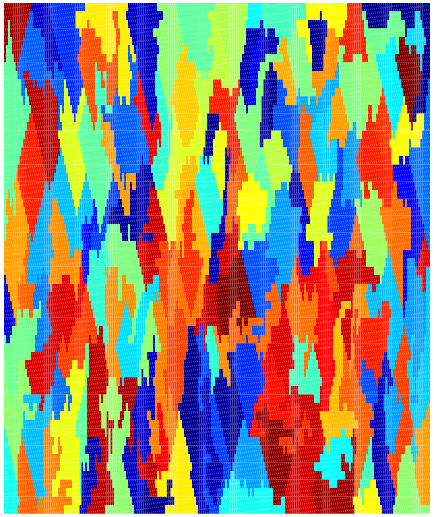
Numerical results



Some issues: Multiscale well modeling,...

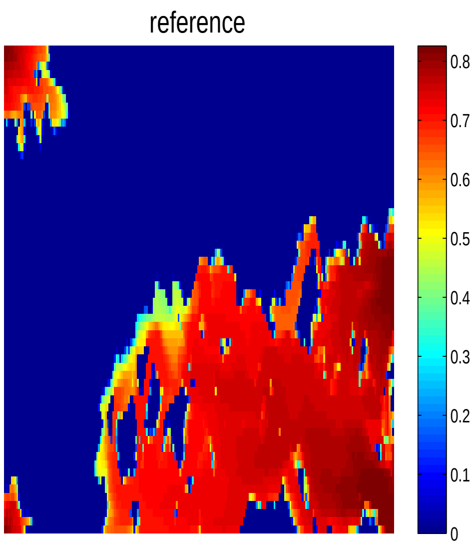
Unstructured coarse grid

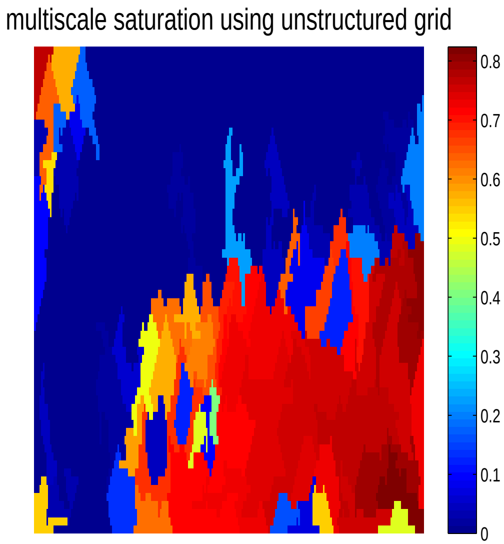


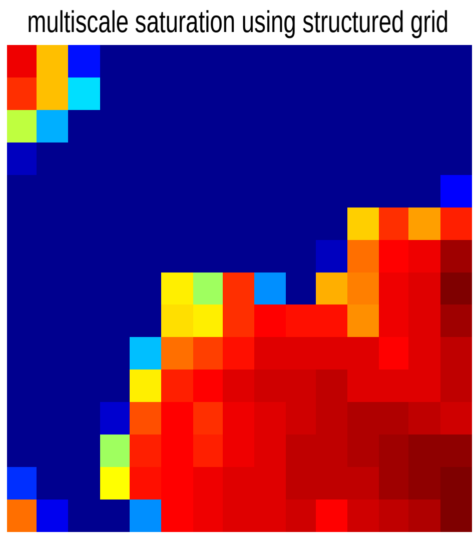


- Geological models are often constructed using highly anistropic corner-point grid at the fine scale. The coarsening of this grid will ultimately lead to unstructured coarse grids.
- The upscaling of saturation equation often requires non-uniform coarsening. Thus, for full upscaled models, we need MsFEMs on unstructured grids.
- The coarse grid is obtained by grouping high and low values of $|v|$ of single-phase flow (J. Aarnes, Hauge, Efendiev, ADWR 2007).

Numerical example





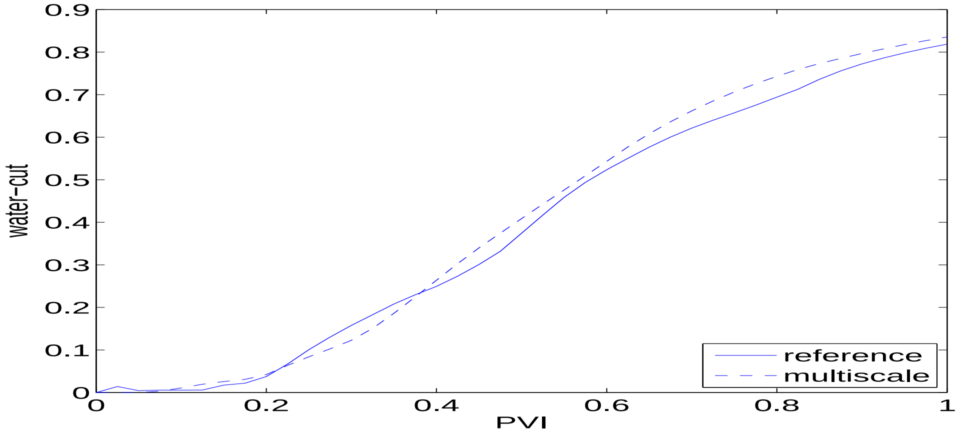


Numerical example

Unstruct. coarse (number of blocks)	sat err. (due to MsFEM)	water-cut err. (due to MsFEM)	sat. err. (total)	sat. err. (struct. gri (total)
180	0.0047	0.0030	0.1936	0.3263 (10x20)
299	0.0040	0.0023	0.1536	0.2831 (15x22)
913	0.0035	0.0012	0.1005	0.2019 (20x44)

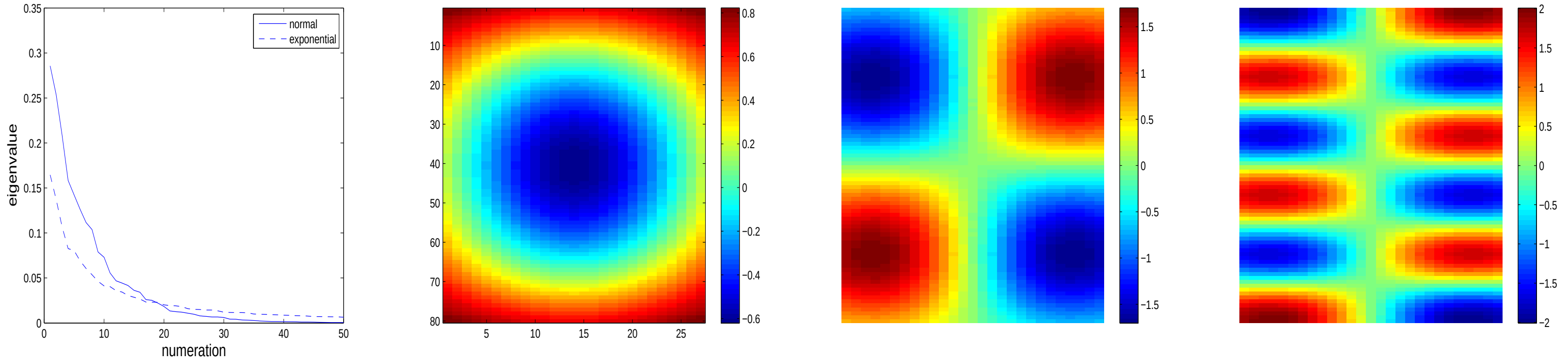
Unstruct. coarse (number of blocks)	sat. err. (total)	sat. err. (struct. grid) (total)
180	0.0097	0.0130 (10x20)
299	0.0080	0.0125 (15x22)
913	0.0062	0.009 (20x44)

Numerical results



Multiscale methods for stochastic flow eqns

- Uncertainties are present in porous media, especially at the fine-scales, $k = k(x, \omega)$.
- E.g., if $Y(x, \omega) = \log(k)$ can be described with 2 point correlation function, $R(x, y) = E(Y(x, \omega)Y(y, \omega))$, then $k(x, \theta) = \exp(\sum_i \theta_i \Psi_i(x))$, where $\Psi_i(x)$ are pre-computed functions based on the covariance matrix and θ_i are iid.



- Our goal is to build *apriori* basis functions that can be used for any realization to solve the flow equations on the coarse grid.
- The pre-computed basis functions are constructed based on selected realizations of the stochastic permeability field.

Ensemble level multiscale finite element method

1. *Generation of coarse grid.*
2. *Construction of multiscale approximation space V_h:*
   - Select N realizations of the permeability field.
   - For each selected realization i;
     - Compute the global information for a particular realization u_i.
     - Compute multiscale basis functions: For each edge e_l^K compute Ψ_l^k.
     - Define $V_h(k_i) = \bigoplus_{l,K} \Psi_l^K$.
   - Define $V_h = \bigoplus_{i=1}^N V_h(k_i)$.
3. *Rapid multiscale computation of velocity solutions for stochastic porous media flow with precomputed basis functions (from Step 2) for each individual arbitrarily chosen realization, e.g., for uncertainty quantification, history matching, etc.*
   - Select a family of arbitrary realizations.
   - For each arbitrarily selected realization, solve coarse-scale equations with pre-computed basis functions.

Numerical setting

- Gaussian:

$$R(x,y) = \sigma^2 \exp\Big(-\frac{|x_1-y_1|^2}{2L_1^2} - \frac{|x_2-y_2|^2}{2L_2^2}\Big).$$

  $k(x,\theta) = \exp(\sum_{i=1}^N \theta_i \Psi_i(x))$. For permeability fields in our numerical experiments, there are about 10-15 terms in KLE. For Gaussian fields, we will be using 1st order sparse interpolation points.

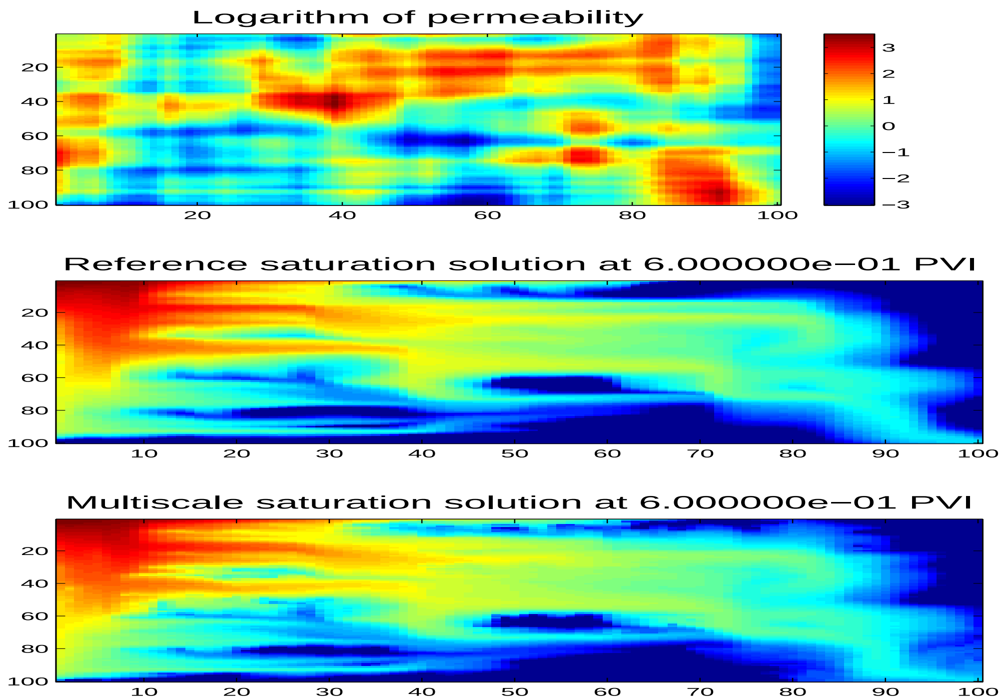
- Exponential variogram

$$R(x,y) = \sigma^2 \exp\Big(-\frac{|x_1-y_1|}{L_1} - \frac{|x_2-y_2|}{L_2}\Big).$$

  The expansion of the permeability in our numerical tests requires many terms (about 300-400 for our problems). We take 10 independent realizations and form basis functions.

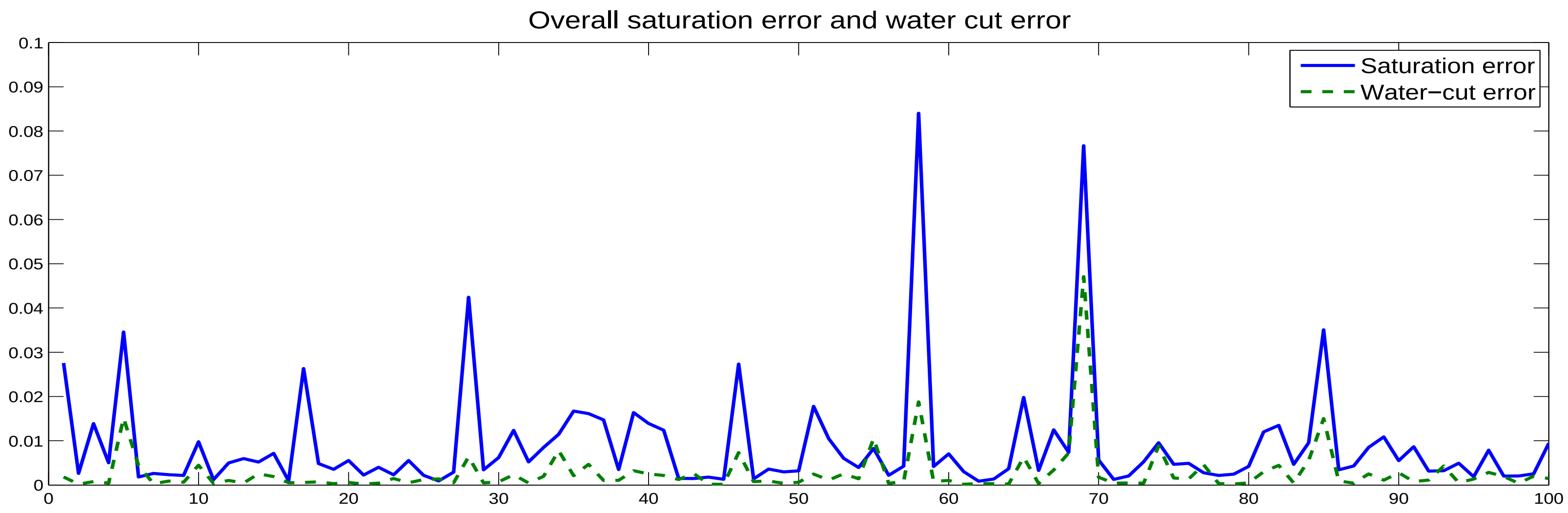
- Fine-scale grid is 100×100. Coarse-scale grid is 5×5.
- $k_{rw} = S^2$, $k_{ro} = (1-S)^2$, $\mu_o/\mu_w = 10$.

Numerical results. Exponential variogram



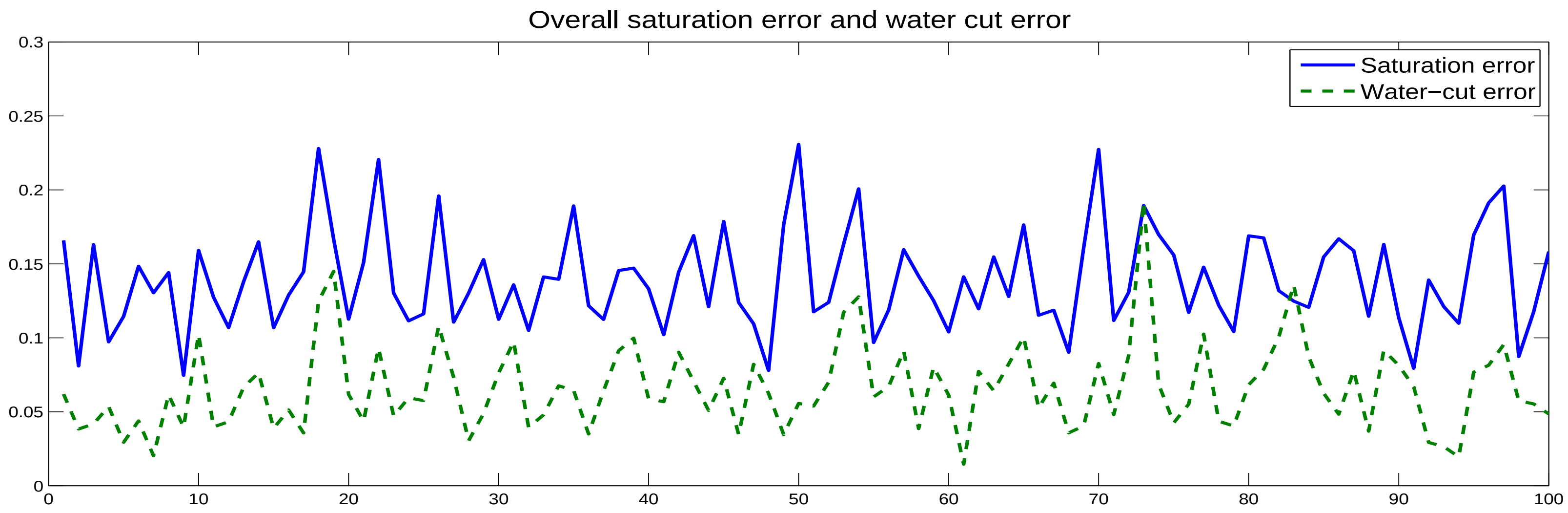
A realization of the permeability and the comparison of the reference saturation field and the saturation profile obtained with multiscale method at $PVI = 0.6$. Exponential case with $L_1 = 0.3$, $L_2 = 0.1$, $\sigma^2 = 2$.

Numerical results. Gaussian variogram



L^2 errors of the saturation field and water-cut errors for 100 randomly chosen realizations. Gaussian field with $l_1 = l_2 = 0.2$, $\sigma^2 = 2$.

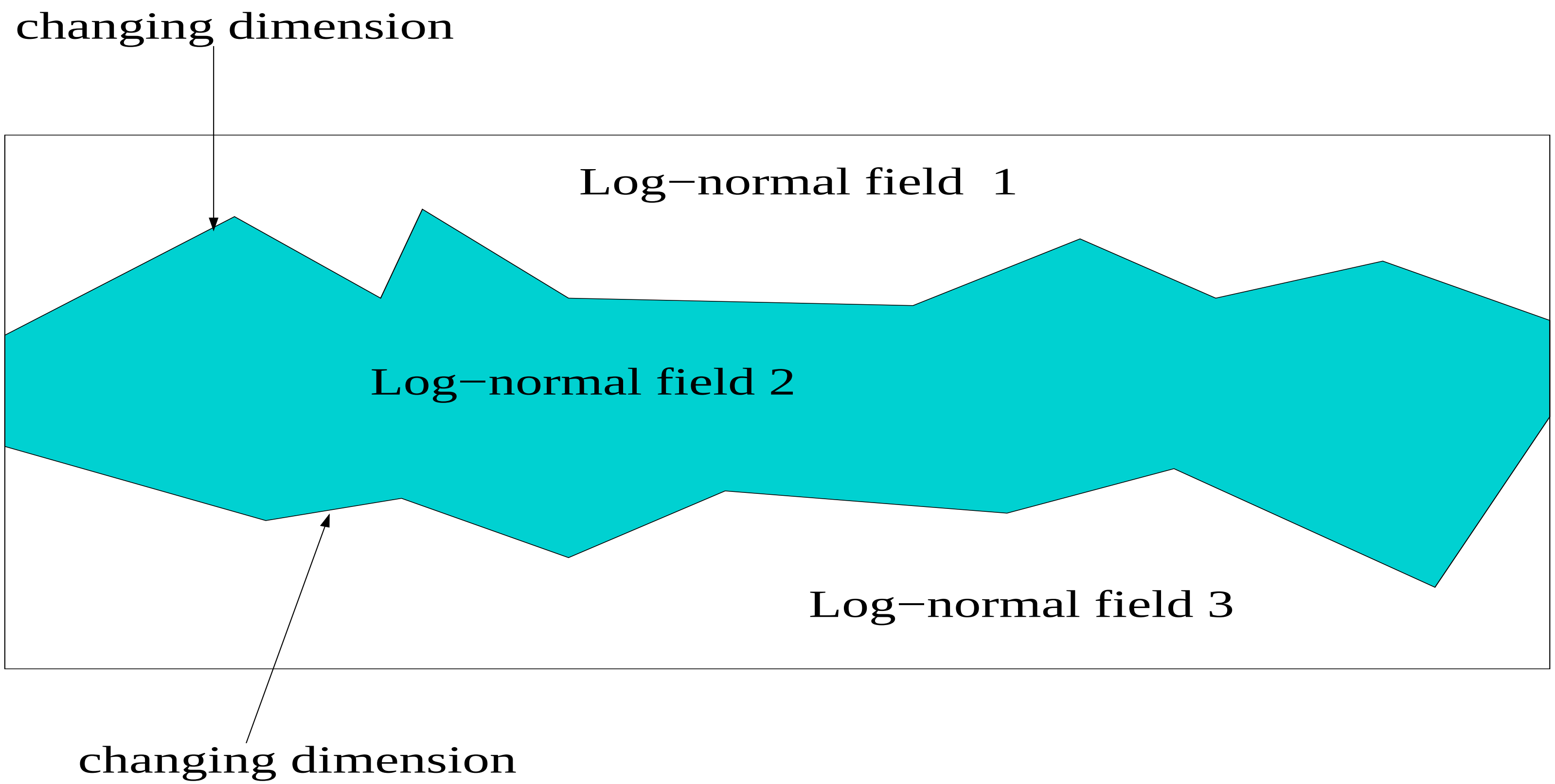
Numerical results. Exponential variogram



L^2 errors of the saturation field and water-cut errors for 100 randomly chosen exponential variogram fields with $l_1 = 0.5$, $l_2 = 0.1$, and $\sigma^2 = 2$.

Extensions

One can also consider hierarchical models representing facies. Stochastic level set equations are used to characterize the “movement” of facies’ boundaries and two-point correlation functions are used to describe the permeability within each facie.



Summary

- An important part of MsFEMs is to construct the multiscale basis functions which can accurately capture the subgrid effects through judicial choice of local boundary conditions.
- MsFEM type methods can be extended to nonlinear problems.
- For problems with strong non-local effects, some type of limited global information is needed to capture the long-range effects.
- The non-local information is based on solutions of some global problems (e.g., oversampling).
- Coarsening.
- The extensions of MsFEMs to stochastic problems. The main idea is to construct multiscale basis functions which can capture the spatial information and heterogeneities.
- Adaptivity in “uncertainty” space can be used to achieve localization.
- Applications.