Experience on Compressible Mantle Convection Code

Eh Tan

3/27/2006

at Purdue University

Compressible Mantle Convection Workshop

Outline

- Penalty method for incompressible flow
  - Continuity and momentum equations
  - Weak form
- Penalty method for compressible flow (failed attempts)
  - Weak form I: asymmetric stiffness matrix
  - Weak form II: symmetric stiffness matrix
- Mixed method for compressible flow
  - Iterative algorithm

Incompressible flow

$$u_{i,i} = 0$$

$$-p_{,i} + \sigma_{ij,j} = \Delta\rho g Ra \delta_{i2}$$

$$\sigma_{ij} = \eta(u_{j,i} + u_{i,j})$$

Penalty method

Let $$p = \lambda u_{j,j}$$

$$\int w_i p_{,i} d\Omega = \int w_i (\lambda u_{j,j})_{,i} d\Omega$$

$$= \cancel{\lambda \oint n_i w_i u_{j,j} d\Gamma} - \lambda \int w_{i,i} u_{j,j} d\Omega$$

$$= -\lambda \int w_{i,i} u_{j,j} d\Omega$$

symmetric

Compressible flow

$$(\rho_r u_i)_{,i} = 0$$

$$-p_{,i} + \sigma_{ij,j} = \Delta\rho g R a \delta_{i2}$$

$$\sigma_{ij} = \eta(u_{j,i} + u_{i,j} - \frac{2}{3} u_{k,k}\delta_{ij})$$

Penalty method I

Let $$p = \lambda(\rho_r u_j)_{,j}$$

$$\int w_i p_{,i} d\Omega = \int w_i \left(\lambda(\rho_r u_j)_{,j}\right)_{,i} d\Omega$$

$$= \cancel{\lambda \oint n_i w_i (\rho_r u_j)_{,j} d\Gamma} - \lambda \int w_{i,i} (\rho_r u_j)_{,j} d\Omega$$

$$= -\lambda \int (\rho_r w_{i,i} u_{j,j} - \rho_{r,j} w_{i,i} u_j) d\Omega$$

symmetric

symmetric

Results

- Asymmetric stiffness matrix
  - solved by LAPACK solver
- Recovering p is straightforward
- Benchmark:
  - stress field is inaccurate

Penalty method II

Let $$p_{,i} = \lambda \rho_r (\rho_r u_j)_{,ji}$$

$$\int w_i p_{,i} d\Omega = \int w_i \lambda \rho_r (\rho_r u_j)_{,ji} d\Omega$$

$$= \cancel{\lambda \oint n_i w_i \rho_r (\rho_r u_j)_{,j} d\Gamma}$$

$$- \lambda \int (\rho_r w_i)_{,i} (\rho_r u_j)_{,j} d\Omega$$

$$= -\lambda \int (\rho_r w_i)_{,i} (\rho_r u_j)_{,j} d\Omega$$

symmetric

Results

- Symmetric stiffness matrix
  - solved by ConMan's original solver
- Recovering *p* is difficult
- Benchmark:
  - accurate in constant η case
  - inaccurate in depth-dep. η case

Iterative algorithm

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_r}\frac{\partial \rho_r}{\partial z} u_z = 0$$

$$\begin{bmatrix} K & G \\ G^T + C_\rho & 0 \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

asymmetric matrix equation

Results

- Asymmetric stiffness matrix
  - solved by Uzawa algorithm using BiCGstab (very similar to the method of Dave and Luis)
- Good accuracy in *v*, *p*, σ
- Dissipative heating term
  - error amplified because of $A^2 - B^2$
  - element-level σ suffers from pressure mode oscillation
  - node-level σ (interpoated from element-level σ) is better

Lessons learnt

- Penalty method + non-iterative algorithm is difficult to be accurate for ConMan
  - because of bilinear-velocity, constant-pressure element?
  - using higher order element (e.g. enriched element [Fortin, 1981])?
- Mixed method + iterative algorithm is accurate for ConMan
  - maybe will work for CitCom too