Implementing the compressible Anelastic approximation in a mantle convection code

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Technical overview

- Experience with spectral 3-D spherical code (direct solver) from Gary Glatzmaier helped, but details quite different:
- Primitive variables (velocity & pressure)
- Velocity components (3) & pressure staggered so that finite-difference derivatives always involve adjacent components
- 3-D variable viscosity and (potentially) other properties

The Anelastic approximation: A straightforward step from the Boussinesq approximation

- Boussinesq: $\nabla \cdot \vec{v} = 0$
  - ρ constant except in buoyancy term
- Anelastic $\nabla \cdot (\overline{\rho}\vec{v}) = 0$
  - $\rho = \overline{\rho}(z)$ except in buoyancy term
  - Viscous dissipation & adiabatic heating terms added to energy equation
  - Div(v) term added in normal stresses
  - Properties like expansivity, conductivity can vary (as they can in the Boussinesq approx.)

Implementation: Continuity

- Finite volume discretized equation requires simple modification: e.g., in 2-D

Boussinesq:

$$\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta z} = 0$$

Anelastic:

$$\overline{\rho}_j \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{\overline{\rho}_{j+1/2} v_{i,j+1/2} - \overline{\rho}_{j-1/2} v_{i,j-1/2}}{\Delta z} = 0$$

Where i=x index, j=y index, u,v are x & y velocities, deltas are grid spacings, velocities are staggered so that they lie at desired locations

Implementation: Energy equation

- Viscous dissipation and adiabatic heating/cooling terms can be treated explicitly (wrt time) and do not affect the stability of the numerical solution scheme, just as diffusion and advection can be treated explicitly.
- Calculated using standard finite difference/finite volume discretization

Implementation: Momentum eqn.

- Normal stresses gain a div(v) term:

$$\tau_{ij} = 2\eta\left(\dot{e}_{ij} - \tfrac{1}{3} e_{kk}\delta_{ij}\right) = 2\eta\left(\dot{e}_{ij} - \tfrac{1}{3}\delta_{ij}\nabla\cdot\vec{v}\right)$$

- If simply implemented this can destabilize iterations towards a velocity-pressure solution!
  - Because div(v) is inaccurate during iterations
  - …and we take the divergence of this term!
- Fix: Use continuity equation to re-express this in terms of vertical velocity (next slide)

Re-express div(v) in stress eq.

Continuity: $$\nabla \cdot (\bar{\rho}\vec{v}) = 0 = \bar{\rho}\nabla \cdot \vec{v} + \vec{v} \cdot \nabla\bar{\rho}$$

Hence: $$\nabla \cdot \vec{v} = -\frac{\vec{v} \cdot \nabla\bar{\rho}}{\bar{\rho}}$$

If $\bar{\rho}(z)$ $$\nabla \cdot \vec{v} = -v_z \frac{\partial\bar{\rho}}{\partial z}\frac{1}{\bar{\rho}} = -v_z \frac{\partial \ln\bar{\rho}}{\partial z}$$

Finally: $$\tau_{ij} = 2\eta\left(\dot{e}_{ij} + \tfrac{1}{3}\delta_{ij}v_z \frac{\partial \ln\bar{\rho}}{\partial z}\right)$$

- Iterations converge stably with this form
- This was the only major problem in implementation!

Reference state, i.e., properties(z)

- In Stag3d, constructed self-consistently using some simple thermodynamic relations, typically from papers that were recent in the early 1990s (details in 1996 JGR).
- Problem arises in fitting upper mantle: real profile of density (& probably other properties) is strongly influenced by complex multi-component phase changes. Also, temperature-dependence is important for thermal conductivity, introducing lateral variations.
- Perhaps best to either
  - Fit PREM, or better:
  - Calculate density etc. as a function of T, p and “composition” (e.g., 5 main oxides) using a full thermodynamic database approach, minimizing free energy,… (so far implemented by Connelly, Matas & Ricard)

Enhancements: Phase changes introduce complexity

- 2-component (olivine, pyroxene) each with its own phase changes: A different reference state for each phase (4 olivine, 5 pyroxene).
- Reference density depends on phase=>lateral as well as vertical variations
- Advecting potential temperature rather than actual temperature automatically accounts for adiabatic heating/cooling and phase change latent heat.

Conclusions

- Changing from Boussinesq to compressible anelastic was straightforward, the main problem being finding a stable treatment of div(v) in the stress terms in the momentum equations
- Many additional stages of complexity/realism are possible so approaches should be flexible.