

Poroelasticity in PyLith

Robert Walker, Matt Knepley

Computer Science and Engineering
University at Buffalo

CIG Seminar
May 12th, 2022



Outline

1

Introduction

- Pylith Refresher

2

Motivation

3

Implementation

4

Results

5

Future Expansion

Outline

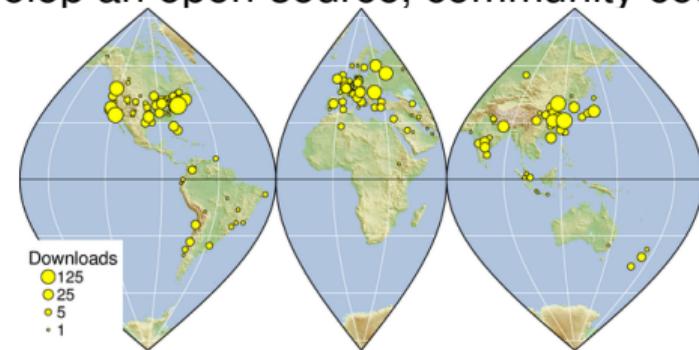
1

- Introduction
 - Pylith Refresher

PyLith

A modern, community-driven code for crustal deformation modeling

- Developers
 - Brad Aagaard (USGS)
 - Matthew Knepley (SUNY Buffalo)
 - Charles Williams (GNS Science)
- Combined dynamic modeling capabilities of EqSim (Aagaard) with the quasi-static modeling capabilities of Tecton (Williams)
- Use modern software engineering to develop an open-source, community code
 - Modular design
 - Testing
 - Documentation
 - Distribution
- PyLith v1.0 was released in 2007



PyLith

- Multiple problems

- Dynamic rupture
- Quasi-static relaxation
- Numerical Green functions
- Earthquake cycle

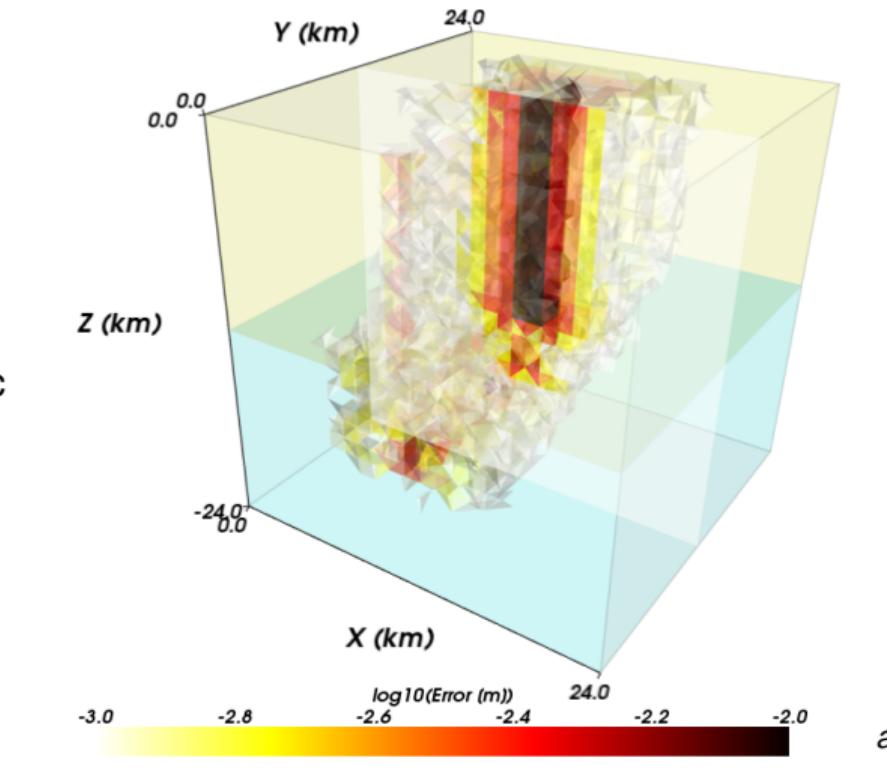
- Multiple models

- Nonlinear visco-plastic
- Finite deformation
- Fault constitutive models

- Multiple meshes

- 1D, 2D, 3D
- Hex and tet meshes
- Refinement and fault insertion

- Scalable, parallel solvers



^aAagaard, Knepley, and Williams 2013

PyLith v3 - Physics Capabilities

- **Governing Equations**

- Quasistatic and dynamic elasticity
- Quasistatic incompressible elasticity
- Quasistatic and dynamic poroelasticity

- **Bulk Rheologies**

- Isotropic, linear elasticity (all governing equations)
- Isotropic, linear poroelasticity
- Viscoelasticity (quasistatic elasticity)

- **Fault**

- Prescribed slip for quasistatic and dynamic problems

- **Boundary Conditions**

- Dirichlet
- Neumann
- Absorbing

Outline

1 Introduction

2 Motivation

- Poroelasticity
- Geophysical Examples
- Problem Description

3 Implementation

4 Results

5 Future Expansion

Outline

2

Motivation

- Poroelasticity
- Geophysical Examples
- Problem Description

What is a Porous Material?



A solid, with holes....

What is Poroelasticity?

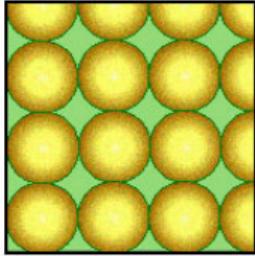
Poroelasticity is the study of the interaction between fluid flow and solid deformation in a porous medium.



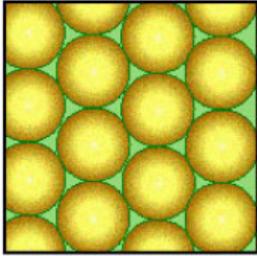
- ➊ Sand
- ➋ Sandstone
- ➌ Volcanic rock
- ➍ Fractured rock
- ➎ Pervious concrete
- ➏ Polyurethane foam
- ➐ Metal foam
- ➑ Bone
- ➒ Articular cartilage
- ➓ Nanoporous alumina

A Solid, With Holes - Porosity

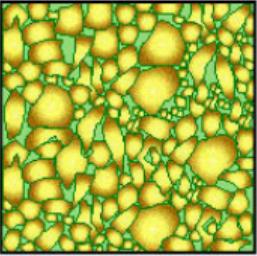
Types Of Pore Distribution



Uniform Spheres
with Cubic Packing



Uniform Spheres with
Rhombohedral Packing



Non-Uniform Particles
with Non-Uniform Packing
of Typical Sandstone
Reservoir Rock

Porosity = 47.6%
(for any Sphere Radius)

Porosity = 25.96%
(for any Sphere Radius)

Porosity \approx 5-25%

INCREASING ENERGY IN THE DEPOSITIONAL ENVIRONMENT



BOUNDSTONE
REEF
OR OTHER
TYPE OF
ORGANIC
BUILDUP



GRAINSTONE
SAND SIZED
GRAINS
WITHOUT
A MUD
MATRIX



PACKSTONE
SAND SIZED
GRAINS
IN CONTACT
FORMING A
GRAIN FRAMEWORK
WITH A MATRIX
OF MORE
THAN 50% MUD

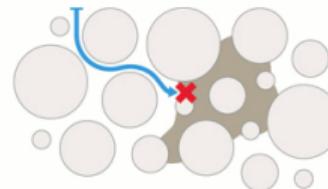
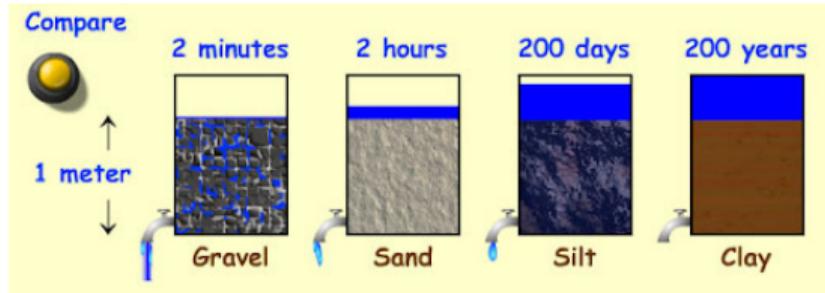


WACKESTONE
SAND SIZED
GRAINS
"FLOATING"
IN A MATRIX
OF MUD



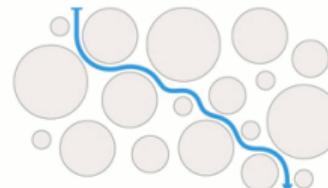
MUDSTONE
<10%
GRAINS

A Solid, With Holes - Permeability



POOR PERMEABILITY

Cement blocks the pores, so the pores are not connected.



GOOD PERMEABILITY

The pores are connected.

Outline

2

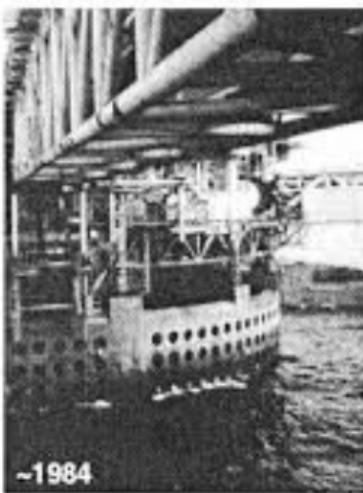
Motivation

- Poroelasticity
- Geophysical Examples
- Problem Description

Subsidence



~1973



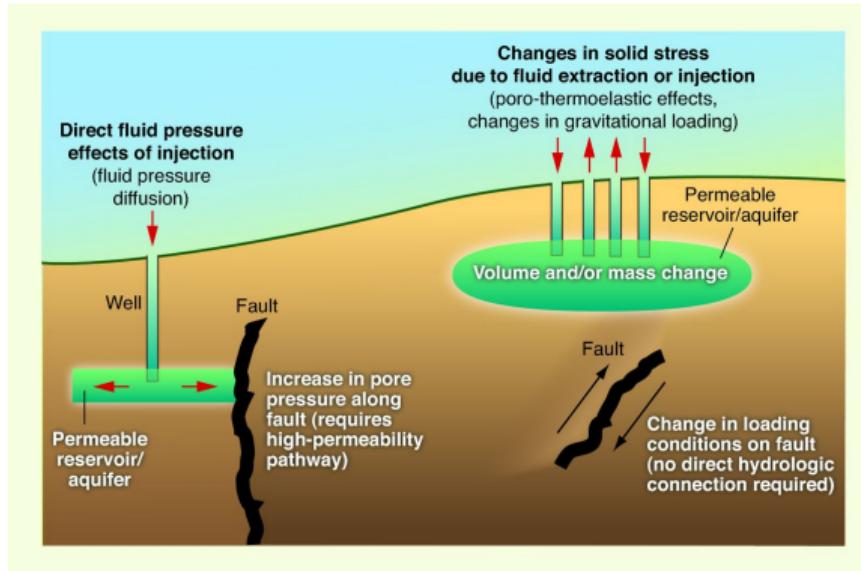
~1984

Ekofisk Field, North Sea

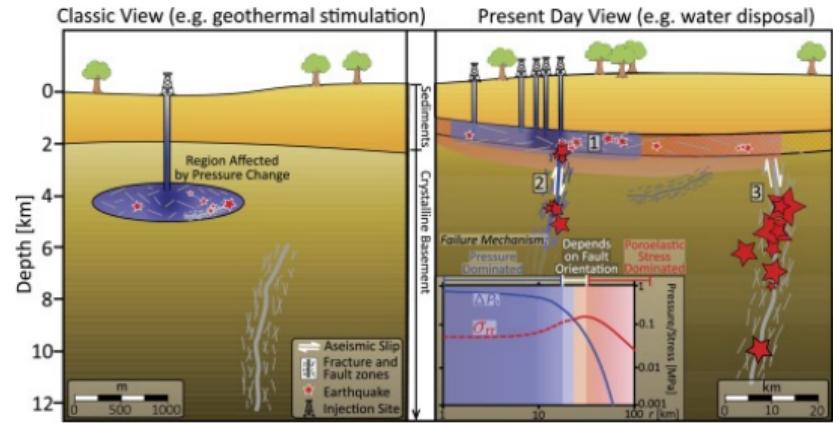


Wilmington Oil Field, California

Induced Seismicity



Ellsworth 2013



Goebel et al. 2017

Outline

2

Motivation

- Poroelasticity
- Geophysical Examples
- Problem Description

Biot Formulation (Biot 1941)

1 Conservation of Momentum

$$\rho_b \ddot{\vec{u}} = \vec{f}(t) + \nabla \cdot \boldsymbol{\sigma}(\vec{u}, p)$$

- Drawn directly from linear elasticity, with stress tensor modified to account for fluid pressure: $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\epsilon} - \alpha \mathbf{I} p$
- Bulk density defined as $\rho_b = (1 - \phi) \rho_s + \phi \rho_f$

2 Conservation of Mass

$$\dot{\zeta} + \nabla \cdot \vec{q}(p) = \gamma(\vec{x}, t)$$

- Specific discharge is defined by Darcy's Law: $\vec{q} = -\frac{\mathbf{k}}{\mu_f} \cdot (\nabla p - \vec{f}_f)$.
- Variation of fluid content is defined as $\zeta = \alpha \epsilon_v + \frac{p}{M}$

Two Coupling Terms

- Biot Coefficient

$$\alpha = 1 - \frac{K_{dr}}{K_s}$$

- Biot Modulus

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{\alpha - \phi}{K_s},$$

Governing Assumptions

for Quasistatic, Isotropic Linear Poroelasticity

- Infinitesimal strain formulation
- Undrained condition
- Linear elastic solid matrix
- Slightly compressible fluid
- Inertial term is negligible ($\rho_b \ddot{\vec{u}} = 0$)
- Solid phase mass is constant, fluid phase mass is conserved.
- We include a volumetric strain relation,

$$\nabla \cdot \vec{u} = \epsilon_v$$

and the volumetric strain term, ϵ_v , to the solution vector to aid with stability close to incompressibility.

Outline

1 Introduction

2 Motivation

3 Implementation

- Multiphysics
- Rheology
- Kernel Implementation
- State Variables

4 Results

5 Future Expansion

Outline

3

Implementation

- Multiphysics
- Rheology
- Kernel Implementation
- State Variables

PyLith v3

Major rewrite of PyLith started in 2016

Feature	PyLith v2	PyLith v3
Governing eqn	Hardwired (elasticity)	Flexible (elasticity, incompressible elasticity, poroelasticity)
Temporal discretization	Backward Euler, Newmark (central difference)	PETSc TS (primarily Runge Kutta)
Spatial discretization	Hardwired (1st order)	Flexible (tested up to 4th order)
Finite-element definition	PyLith	PETSc

Aside: Finite-Element Method

Strong form to weak form

Solve governing equation in integrated sense:

$$\int_{\Omega} \psi_{trial} \cdot PDE \, d\Omega = 0, \quad (1)$$

by minimizing the error with respect to the unknown coefficients.

This leads to equations of the form:

$$\int_{\Omega} \psi_{trial} \cdot f_0(x, t) + \nabla \psi_{trial} \cdot f_1(x, t) \, d\Omega = 0. \quad (2)$$

Multiphysics Implicit Formulation

Implement physics using point-wise integration functions

We want to solve equations in which the weak form can be expressed as

$$\vec{F}(t, s, \dot{s}) = \vec{G}(t, s)$$

$$F(t, s, \dot{s}) = \vec{0}$$

$$\vec{F}(t, s, \dot{s}) = \cancel{\vec{G}(t, s)}^{\vec{0}}$$

$$\vec{s}(t_0) = s_0$$

Thus, all terms are shifted to the Left Hand Side.

Multiphysics Implicit Formulation

Implement physics using point-wise integration functions

We want to solve equations in which the weak form can be expressed as

$$\vec{F}(t, s, \dot{s}) = \vec{G}(t, s)$$

$$F(t, s, \dot{s}) = \vec{0}$$

$$\vec{F}(t, s, \dot{s}) = \cancel{\vec{G}(t, s)}^{\vec{0}}$$

$$\vec{s}(t_0) = s_0$$

Thus, all terms are shifted to the Left Hand Side.

Using the finite-element method and divergence theorem, we cast the weak form into

$$\int_{\Omega} \vec{\psi}_{trial} \cdot \vec{f}_0(t, s, \dot{s}) + \nabla \vec{\psi}_{trial} : \vec{f}_1(t, s, \dot{s}) d\Omega =$$

$$\int_{\Omega} \vec{\psi}_{trial} \cdot \cancel{\vec{g}_0(t, s)}^{\vec{0}} + \nabla \vec{\psi}_{trial} : \cancel{\vec{g}_1(t, s)}^{\vec{0}} d\Omega$$

Multiphysics Formulation: Poroelasticity

Formulation shown for quasistatic case without fault

Strong form

$$\vec{s}^T = (\vec{u} \quad p \quad \epsilon_v), \quad (3)$$

$$\vec{f}(t) + \nabla \cdot \boldsymbol{\sigma}(\vec{u}, p) = \vec{0} \text{ in } \Omega, \quad (4)$$

$$\frac{\partial \zeta(\vec{u}, p)}{\partial t} - \gamma(\vec{x}, t) + \nabla \cdot \vec{q}(p) = 0 \text{ in } \Omega, \quad (5)$$

$$\nabla \cdot \vec{u} - \epsilon_v = 0 \text{ in } \Omega, \quad (6)$$

$$\boldsymbol{\sigma} \cdot \vec{n} = \vec{\tau}(\vec{x}, t) \text{ on } \Gamma_\tau, \quad (7)$$

$$\vec{u} = \vec{u}_0(\vec{x}, t) \text{ on } \Gamma_u, \quad (8)$$

$$\vec{q} \cdot \vec{n} = q_0(\vec{x}, t) \text{ on } \Gamma_q, \text{ and} \quad (9)$$

$$p = p_0(\vec{x}, t) \text{ on } \Gamma_p. \quad (10)$$

Multiphysics Formulation: Poroelasticity

Formulation shown for quasistatic case without fault

Weak form

$$\int_{\Omega} \vec{\psi}_{trial}^u \cdot \underbrace{\vec{f}(\vec{x}, t)}_{f_0^u} + \nabla \vec{\psi}_{trial}^u : \underbrace{-\boldsymbol{\sigma}(\vec{u}, p_f)}_{f_1^u} d\Omega + \int_{\Gamma_\tau} \vec{\psi}_{trial}^u \cdot \underbrace{\vec{\tau}(\vec{x}, t)}_{f_0^u} d\Gamma = 0, \quad (3)$$

$$\int_{\Omega} \psi_{trial}^p \underbrace{\left(\frac{\partial \zeta(\vec{u}, p_f)}{\partial t} - \gamma(\vec{x}, t) \right)}_{f_0^p} d\Omega + \nabla \psi_{trial}^p \cdot \underbrace{-\vec{q}(p_f)}_{f_1^p} d\Omega + \int_{\Gamma_q} \psi_{trial}^p \underbrace{q_0(\vec{x}, t)}_{f_0^p} d\Gamma = 0, \quad (4)$$

$$\int_{\Omega} \psi_{trial}^{\epsilon_v} \cdot \underbrace{(\nabla \cdot \vec{u} - \epsilon_v)}_{f_0^{\epsilon_v}} d\Omega = 0. \quad (5)$$

Summary of Multiphysics Implementation

We decouple the finite-element definition from the weak form equation, using pointwise functions that look like the PDE.

Each material and boundary condition contribute pointwise functions.

Flexibility The cell traversal, handled by PETSc, accommodates arbitrary cell shapes. The problem can be posed in any spatial dimension with an arbitrary number of physical fields.

Extensibility PETSc needs to maintain only a single method, easing language transitions (CUDA, OpenCL). A new discretization scheme could be enabled in a single place in the code.

Efficiency Only a few PETSc routines need to be optimized. The application scientist is no longer responsible for proper vectorization, tiling, and other traversal optimization.

Outline

3

Implementation

- Multiphysics
- Rheology
- Kernel Implementation
- State Variables

Rheology Concept

Elasticity and Rheologies

Material	Bulk Rheology	Description
Elasticity	IsotropicLinearElasticity	Isotropic, linear elasticity
	IsotropicLinearMaxwell	Isotropic, linear Maxwell viscoelasticity
	IsotropicLinearGenMaxwell	Isotropic, generalized Maxwell viscoelasticity
	IsotropicPowerLaw	Isotropic, power-law viscoelasticity
	IsotropicDruckerPrager	Isotropic, Drucker-Prager elastoplasticity

Poroelasticity has ONE rheology (at the moment).

Auxiliary Fields

for Quasistatic Linear Isotropic Poroelasticity

Origin	Variable	Description	Position	Notes
Material	ρ_b	Rock Density	0	
	ρ_f	Fluid Density	1	
	μ_f	Fluid Viscosity	2	
	ϕ	Porosity	3	
	\vec{f}_b	Body Force	+1	
	\vec{g}	Gravity	+1	
	γ	Fluid Source	+1	
Rheology	σ_R	Reference Stress	NumAux - 7	
	ϵ_R	Reference Strain	NumAux - 6	
	G	Shear Modulus	NumAux - 5	
	K_d	Drained Bulk Modulus	NumAux - 4	
	α	Biot Coefficient	NumAux - 3	
	M	Biot Modulus	NumAux - 2	$\frac{K_f}{\phi} + \frac{K_s}{\alpha-\phi}$
	k	Permeability	NumAux - 1	
Input	K_s	Solid Grain Bulk Modulus	-	
	K_f	Fluid Bulk Modulus	-	

Residuals

for Quasistatic, Isotropic Linear Poroelasticity

$$F^u(t, s, \dot{s}) = \int_{\Omega} \vec{\psi}_{trial}^u \cdot \underbrace{\vec{f}(\vec{x}, t)}_{\vec{f}_0^u} + \nabla \vec{\psi}_{trial}^u : \underbrace{-\boldsymbol{\sigma}(\vec{u}, p_f)}_{\vec{f}_1^u} d\Omega + \int_{\Gamma_\tau} \vec{\psi}_{trial}^u \cdot \underbrace{\vec{\tau}(\vec{x}, t)}_{\vec{f}_0^u} d\Gamma,$$

$$F^p(t, s, \dot{s}) = \int_{\Omega} \underbrace{\psi_{trial}^p \left[\frac{\partial \zeta(\vec{u}, p_f)}{\partial t} - \gamma(\vec{x}, t) \right]}_{f_0^p} + \nabla \psi_{trial}^p \cdot \underbrace{-\vec{q}(p_f)}_{\vec{f}_1^p} d\Omega + \int_{\Gamma_q} \psi_{trial}^p \underbrace{(q_0(\vec{x}, t))}_{f_0^p} d\Gamma,$$

$$F^{\epsilon_v}(t, s, \dot{s}) = \int_{\Omega} \psi_{trial}^{\epsilon_v} \cdot \underbrace{(\nabla \cdot \vec{u} - \epsilon_v)}_{f_0^{\epsilon_v}} d\Omega.$$

Brown refers to Material, Green to Rheology

Jacobians

for Quasistatic, Isotropic Linear Poroelasticity

$$J_F^{uu} = \frac{\partial F^u}{\partial u} + t_{shift} \frac{\partial F^u}{\partial \dot{u}} = \int_{\Omega} \psi_{trial,i,k}^u \underbrace{(-C_{ikjl})}_{J_{f3}^{uu}} \psi_{basis,j,l}^u d\Omega$$

$$J_F^{up} = \frac{\partial F^u}{\partial p} + t_{shift} \frac{\partial F^u}{\partial \dot{p}} = \int_{\Omega} \psi_{trial,i,j}^u \underbrace{(\alpha \delta_{ij})}_{J_{f2}^{up}} \psi_{basis}^p d\Omega$$

$$J_F^{u\epsilon_v} = \frac{\partial F^u}{\partial \epsilon_v} + t_{shift} \frac{\partial F^u}{\partial \dot{\epsilon}_v} = \int_{\Omega} \nabla \vec{\psi}_{trial}^u : \frac{\partial}{\partial \epsilon_v} [- (2\mu\epsilon + \lambda \mathbf{I}\epsilon_v - \alpha \mathbf{I}p)] d\Omega = \int_{\Omega} \psi_{trial,i,j}^u \underbrace{(-\lambda \delta_{ij})}_{J_{f2}^{u\epsilon_v}} \psi_{basis}^{\epsilon_v} d\Omega$$

$$J_F^{pp} = \frac{\partial F^p}{\partial p} + t_{shift} \frac{\partial F^p}{\partial \dot{p}} = \int_{\Omega} \psi_{trial,k}^p \underbrace{\left(-\frac{\mathbf{k}}{\mu_f} \delta_{kl} \right)}_{J_{f3}^{pp}} \psi_{basis,l}^p d\Omega + \int_{\Omega} \psi_{trial}^p \underbrace{\left(t_{shift} \frac{1}{M} \right)}_{J_{f0}^{pp}} \psi_{basis}^p d\Omega$$

$$J_F^{p\epsilon_v} = \frac{\partial F^p}{\partial \epsilon_v} + t_{shift} \frac{\partial F^p}{\partial \dot{\epsilon}_v} = \int_{\Omega} \psi_{trial}^p \underbrace{(t_{shift} \alpha)}_{J_{f0}^{p\epsilon_v}} \psi_{basis}^{\epsilon_v} d\Omega$$

$$J_F^{\epsilon_v u} = \frac{\partial F^{\epsilon_v}}{\partial u} + t_{shift} \frac{\partial F^{\epsilon_v}}{\partial \dot{u}} = \int_{\Omega} \psi_{trial}^{\epsilon_v} \nabla \cdot \vec{\psi}_{basis}^u d\Omega = \int_{\Omega} \psi_{basis}^{\epsilon_v} \underbrace{(\delta_{ij})}_{J_{f1}^{\epsilon_v u}} \psi_{basis,i,j}^u d\Omega$$

$$J_F^{\epsilon_v \epsilon_v} = \frac{\partial F^{\epsilon_v}}{\partial \epsilon_v} + t_{shift} \frac{\partial F^{\epsilon_v}}{\partial \dot{\epsilon}_v} = \int_{\Omega} \psi_{basis}^{\epsilon_v} \underbrace{(-1)}_{J_{f0}^{\epsilon_v \epsilon_v}} \psi_{basis}^{\epsilon_v} d\Omega$$

Outline

3

Implementation

- Multiphysics
- Rheology
- Kernel Implementation
- State Variables

Overview

- Plug and play physics kernels
- Removes modeler from low level finite element design
- If you can write these, you can implement new physics

Kernel Inputs

```
/** Kernel interface.  
*  
* @param[in] dim Spatial dimension.  
* @param[in] numS Number of registered subfields in solution field.  
* @param[in] numA Number of registered subfields in auxiliary field.  
* @param[in] sOff Offset of registered subfields in solution field [numS].  
* @param[in] sOff_x Offset of registered subfields in gradient of the solution field  
[numS].  
* @param[in] s Solution field with all subfields.  
* @param[in] s_t Time derivative of solution field.  
* @param[in] s_x Gradient of solution field.  
* @param[in] aOff Offset of registered subfields in auxiliary field [numA]  
* @param[in] aOff_x Offset of registered subfields in gradient of auxiliary field [numA].  
* @param[in] a Auxiliary field with all subfields.  
* @param[in] a_t Time derivative of auxiliary field.  
* @param[in] a_x Gradient of auxiliary field.  
* @param[in] t Time for residual evaluation.  
* @param[in] x Coordinates of point evaluation.  
* @param[in] numConstants Number of registered constants.  
* @param[in] constants Array of registered constants.  
* @param[out] f0 [dim].  
*/
```

Translated to Linear Poroelasticity

$$F^u(t, s, \dot{s}) = \int_{\Omega} \vec{\psi}_{trial}^u \cdot \underbrace{\vec{f}(\vec{x}, t)}_{\vec{f}_0^u} + \nabla \vec{\psi}_{trial}^u : \underbrace{-\boldsymbol{\sigma}(\vec{u}, p_f)}_{\vec{f}_1^u} d\Omega + \int_{\Gamma_\tau} \vec{\psi}_{trial}^u \cdot \underbrace{\vec{\tau}(\vec{x}, t)}_{\vec{f}_0^u} d\Gamma,$$

$$F^p(t, s, \dot{s}) = \int_{\Omega} \left[\psi_{trial}^p \left[\underbrace{\frac{\partial \zeta(\vec{u}, p_f)}{\partial t} - \gamma(\vec{x}, t)}_{\vec{f}_0^p} \right] + \nabla \psi_{trial}^p \cdot \underbrace{-\vec{q}(p_f)}_{\vec{f}_1^p} \right] d\Omega + \int_{\Gamma_q} \psi_{trial}^p \left[\underbrace{q_0(\vec{x}, t)}_{\vec{f}_0^p} \right]$$

$$F^\epsilon(t, s, \dot{s}) = \int_{\Omega} \psi_{trial}^\epsilon \cdot \underbrace{(\nabla \cdot \vec{u} - \epsilon_v)}_{\vec{f}_0^\epsilon} d\Omega.$$

Kernel / Residual Pointwise Function f_1^u

$$-\boldsymbol{\sigma} = - \left(K_{dr} - \frac{2G}{3} \right) \boldsymbol{\epsilon}_v - 2G\boldsymbol{\epsilon} + \alpha \mathbf{I} p$$

```
f1u(...){ ...
    for (PylithInt c = 0; c < _dim; ++c) {
        for (PylithInt d = 0; d < _dim; ++d) {
            f1[c*_dim+d] -= shearModulus * (displacement_x[c*_dim+d] +
                displacement_x[d*_dim+c]);
        } // for
        f1[c*_dim+c] -= (drainedBulkModulus - (2.0*shearModulus)/3.0) * trace_strain;
        f1[c*_dim+c] += biotCoefficient*pressure;
    } // for
} // f1u
```

Kernel / Residual Pointwise Function f_0^p

$$\alpha \dot{\epsilon}_v + \frac{\dot{p}}{M} - \gamma$$

```
f0p(...){ ...  
    f0[0] += s_t ? (biotCoefficient * trace_strain_t) : 0.0;  
    f0[0] += s_t ? (pressure_t / biotModulus) : 0.0;  
} // f0p_implicit
```

Kernel / Residual Pointwise Function \vec{f}_1^p

$$-\vec{q} = + \frac{\mathbf{k}}{\mu_f} \cdot (\nabla p + \rho_f \vec{g})$$

```
f1u(...){ ...
    for (PyLithInt d = 0; d < _dim; ++d) {
        f1[d] += (isotropicPermeability / fluidViscosity) * (pressure_x[d]);
    } // for
} // f1p
```

Outline

3

Implementation

- Multiphysics
- Rheology
- Kernel Implementation
- State Variables

State Variable Update

- Auxiliary variables can be iteratively updated
- Update physics enclosed in kernel

State Variable Update - Porosity

$$\frac{\partial \phi}{\partial t} = (\alpha - \phi) \frac{\partial \epsilon_v}{\partial t} + \frac{(1 - \alpha)(\alpha - \phi)}{K_{dr}} \frac{\partial p}{\partial t}$$

Outline

1 Introduction

2 Motivation

3 Implementation

4 Results

- Benchmarks
- Test Examples

5 Future Expansion

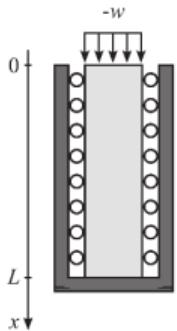
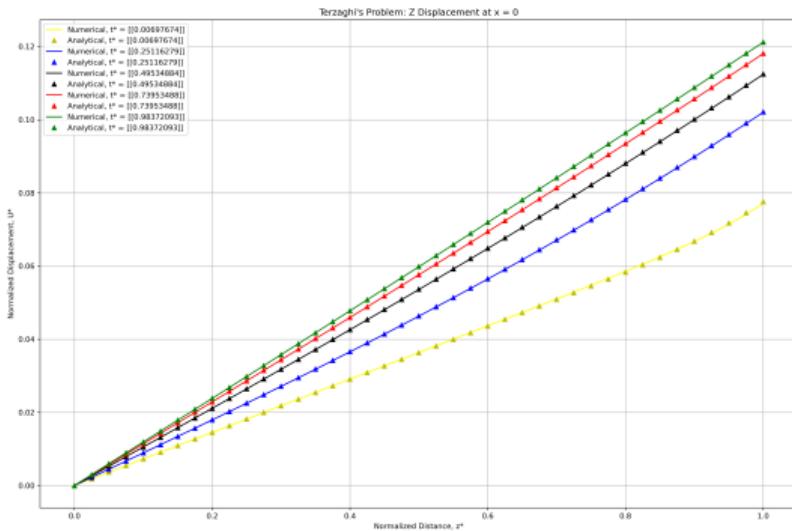
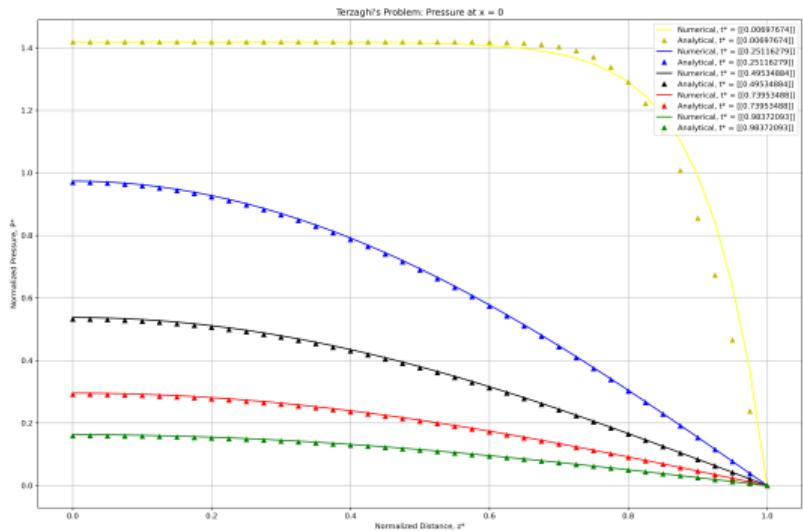
Outline

4

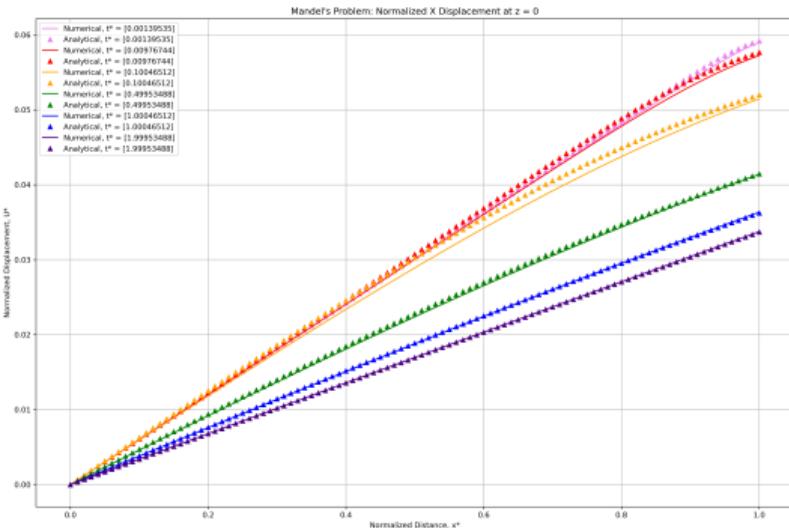
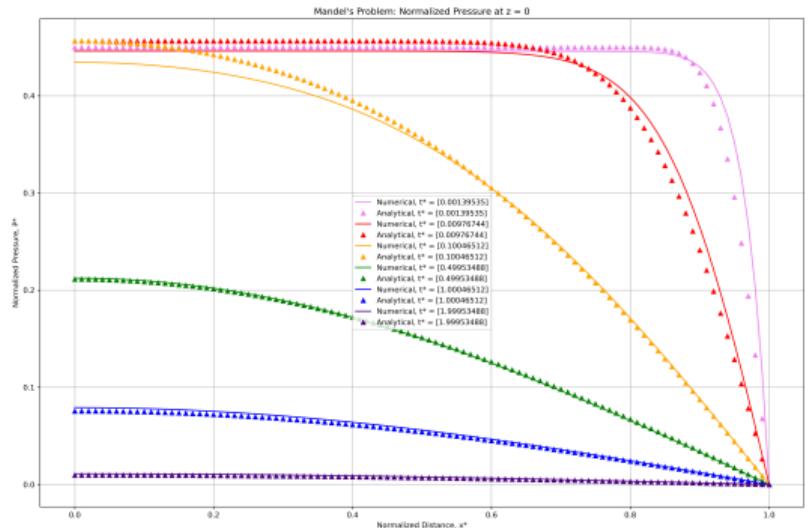
Results

- Benchmarks
- Test Examples

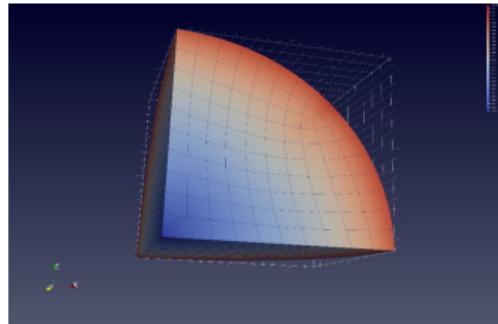
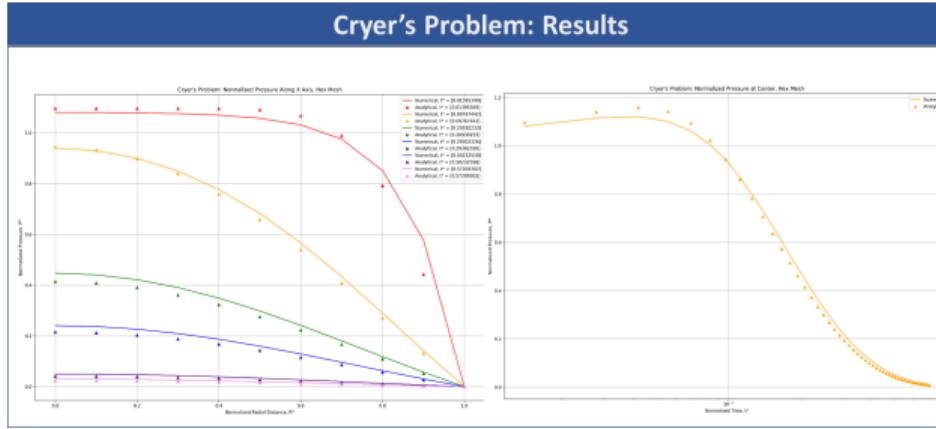
Terzaghi's Problem Test Results



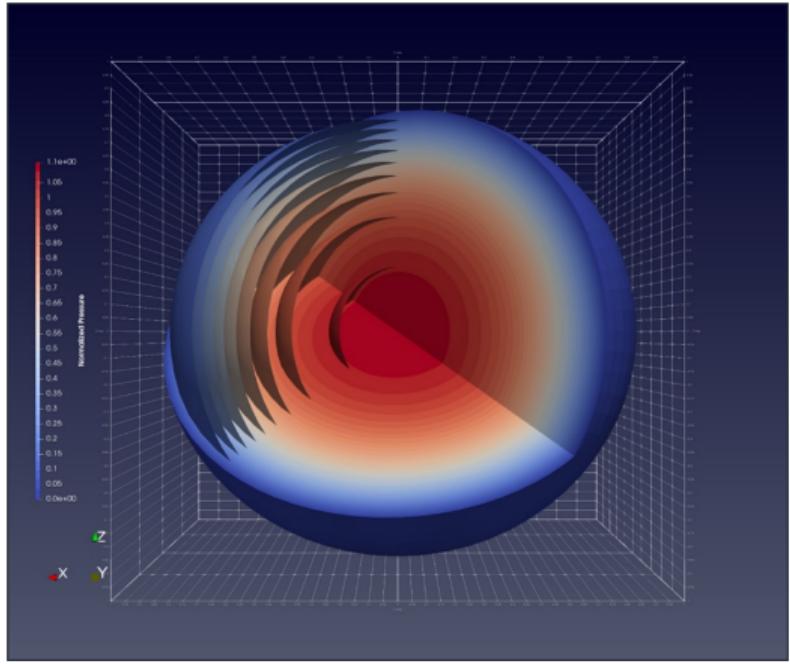
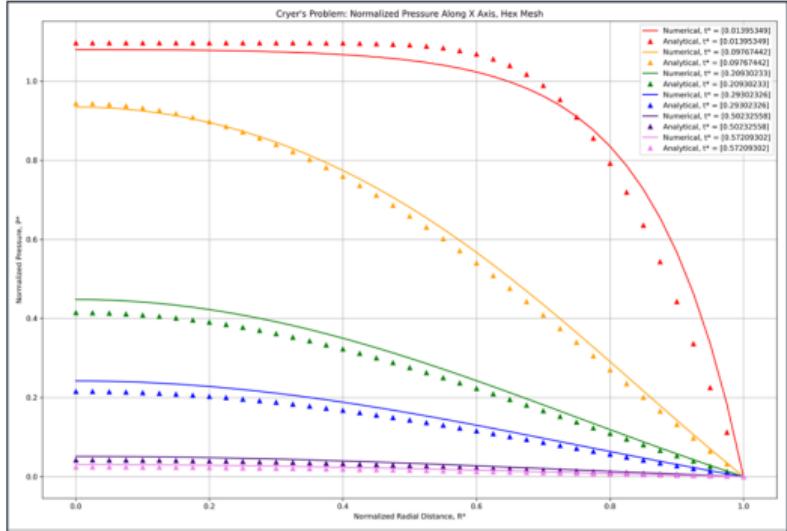
Mandel's Problem Test Results



Cryer's Problem Test Results - Part



Cryer's Problem Test Results - Full



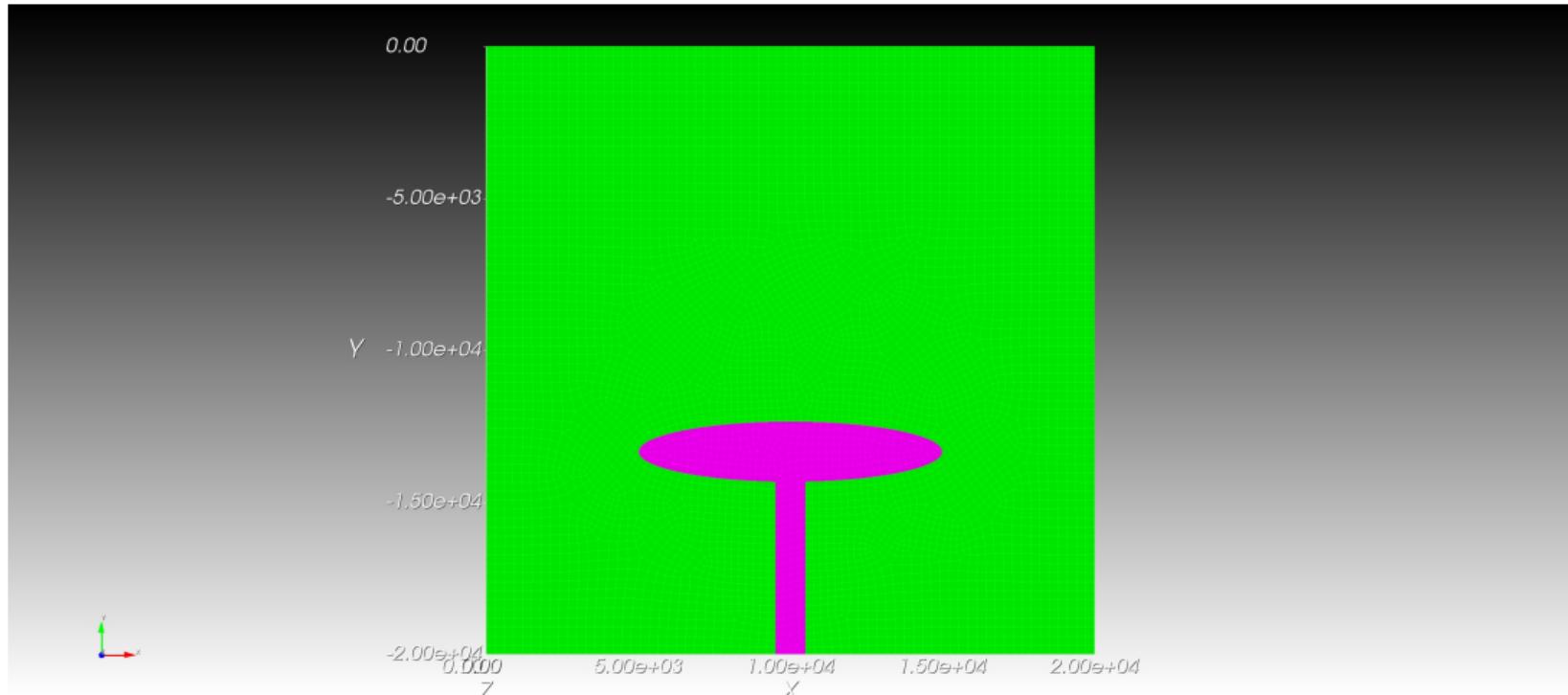
Outline

4

Results

- Benchmarks
- Test Examples

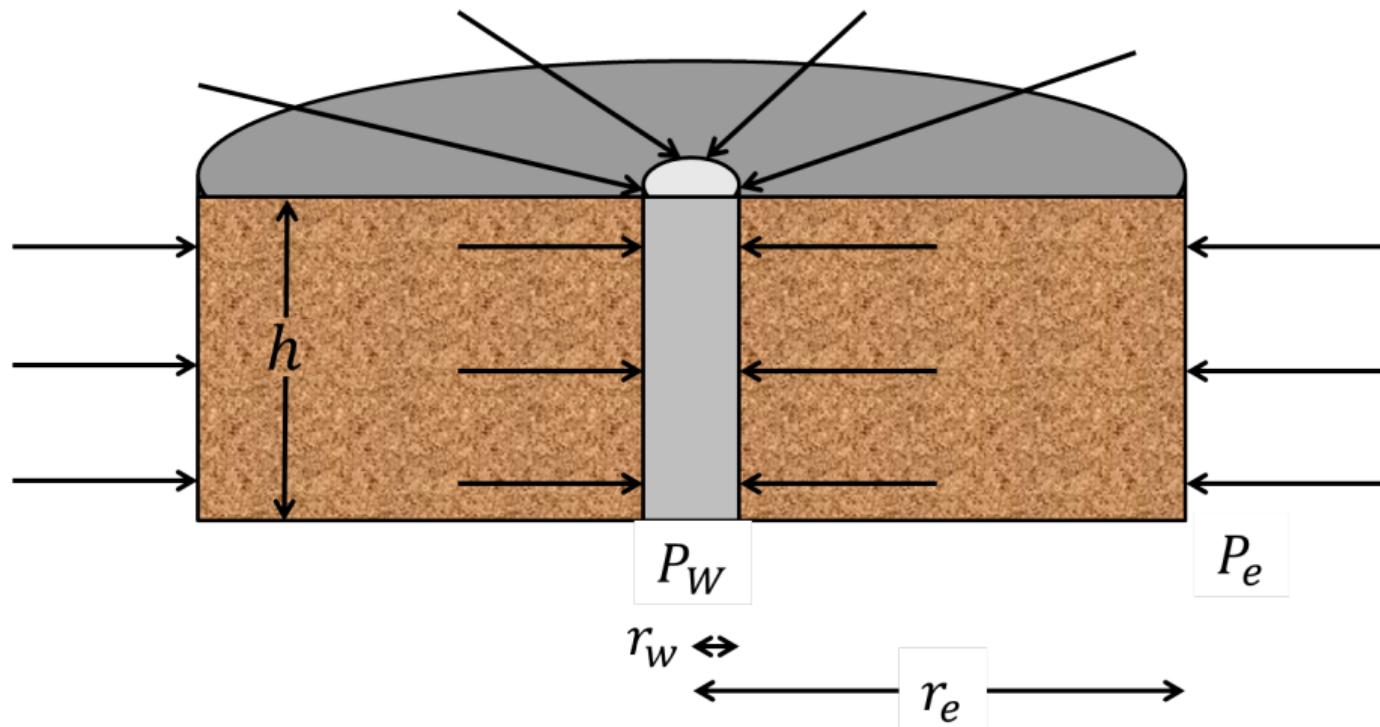
Magma Chamber



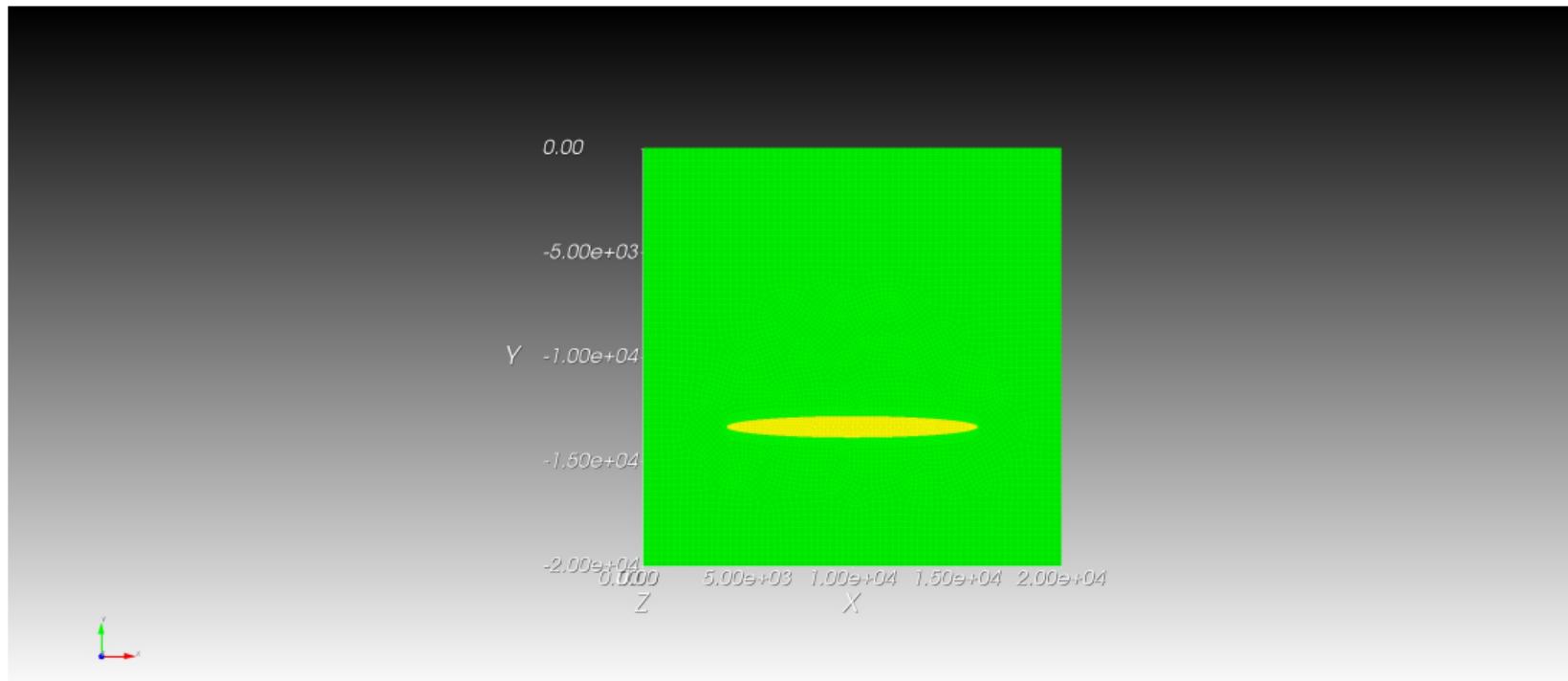
Magma Chamber Inflow

Origin	Variable	Description	Domain	Intrusion
Material	ρ_b	Rock Density	$2500 \frac{kg}{m^3}$	$2500 \frac{kg}{m^3}$
	ρ_f	Fluid Density	$1000 \frac{kg}{m^3}$	$1000 \frac{kg}{m^3}$
	μ_f	Fluid Viscosity	0.001 Pa*s	0.001 Pa*s
	ϕ	Porosity	0.01	0.10
	\vec{f}_b	Body Force	-	-
	\vec{g}	Gravity	-	-
	γ	Fluid Source	-	-
	P_0	Initial Pressure	$5e6 \text{ Pa}$	$5e6 \text{ Pa}$
Rheology	G	Shear Modulus	$6e9 \text{ Pa}$	$6e9 \text{ Pa}$
	K_d	Drained Bulk Modulus	$10e9 \text{ Pa}$	$10e9 \text{ Pa}$
	α	Biot Coefficient	1.0	0.8
	K_f	Fluid Bulk Modulus	$2e9 \text{ Pa}$	$2e9 \text{ Pa}$
	k	Permeability	$1e - 15 \text{ m}^2$	$1e - 13 \text{ m}^2$
Source	P_{in}	Crossflow Pressure	$1e7 \text{ Pa}$	
	l_{in}	Crossflow Boundary Length	1000 m	

Injection "Point" Source



Magma Chamber via Injection



Magma Chamber via Injection

Origin	Variable	Description	Domain	Intrusion
Material	ρ_b	Rock Density	$2500 \frac{kg}{m^3}$	$2500 \frac{kg}{m^3}$
	ρ_f	Fluid Density	$1000 \frac{kg}{m^3}$	$1000 \frac{kg}{m^3}$
	μ_f	Fluid Viscosity	0.001 Pa*s	0.001 Pa*s
	ϕ	Porosity	0.01	0.10
	\vec{f}_b	Body Force	-	-
	\vec{g}	Gravity	-	-
	γ	Fluid Source	-	-
	P_0	Initial Pressure	$5e6 \text{ Pa}$	$5e6 \text{ Pa}$
Rheology	G	Shear Modulus	$6e9 \text{ Pa}$	$6e9 \text{ Pa}$
	K_d	Drained Bulk Modulus	$10e9 \text{ Pa}$	$10e9 \text{ Pa}$
	α	Biot Coefficient	1.0	0.8
	K_f	Fluid Bulk Modulus	$2e9 \text{ Pa}$	$2e9 \text{ Pa}$
	k	Permeability	$1e - 15 \text{ m}^2$	$1e - 14 \text{ m}^2$
Source	P_{wf}	Wellbore Pressure	$10e6 \text{ Pa}$	
	r_w	Wellbore Radius	0.05 m	

Outline

1 Introduction

2 Motivation

3 Implementation

4 Results

5 Future Expansion
• Final Thoughts

Outline

5

- Future Expansion
- Final Thoughts

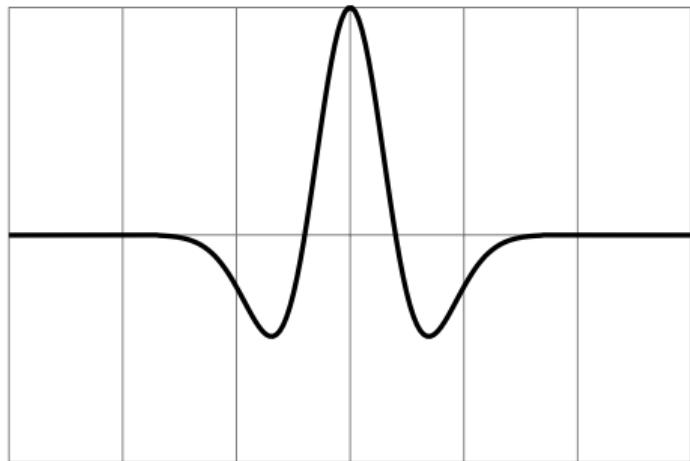
Next Steps

Ordered by Difficulty Level

- Straightforward
 - Clean and test dynamic poroelasticity for functionality
 - Update functions [Additional] for auxiliary variables (k)
 - Conversion scripts for poroelastic benchmark cases (e.g. SPE10)
- Medium
 - Analytical tests for dynamic poroelasticity
 - Well model combined with point source
- More difficult
 - Multiphase model
 - Fully poroelastic faults (Actually not...coming soon!)

Moment Tensor Point Sources

Ricker Wavelet



Dynamic Poroelasticity

Explicit Time Stepping

- Explicit time stepping with the PETSc TS requires $F(t, s, \dot{s}) = \dot{s}$.
- Normally $F(t, s, \dot{s})$ contains the inertial term ($\rho\ddot{u}$).
- Therefore, when using explicit time stepping we transform our equation into the form:

$$\begin{aligned} F^*(t, s, \dot{s}) &= \dot{s} = G^*(t, s) \\ \dot{s} &= M^{-1}G(t, s). \end{aligned}$$

- Terms must be rewritten to ensure that the LHS consists only of time derivatives and coefficients.
- Velocity is introduced as an unknown, again resulting in a three field solution

Dynamic Poroelasticity

Strong Formulation

For compatibility with PETSc TS algorithms, we want to turn the second order equation elasticity equation into two first order equations. We introduce the velocity as a unknown, $\vec{v} = \frac{\partial \vec{u}}{\partial t}$, which leads to a slightly different three field problem,

$$\vec{s}^T = (\vec{u} \quad p \quad \vec{v})$$

$$\frac{\partial \vec{u}}{\partial t} = \vec{v} \text{ in } \Omega$$

$$\frac{1}{M} \frac{\partial p}{\partial t} = \gamma(\vec{x}, t) - \alpha \left(\nabla \cdot \dot{\vec{u}} \right) - \nabla \cdot \vec{q}(p) = 0 \text{ in } \Omega$$

$$\rho_b \frac{\partial \vec{v}}{\partial t} = \vec{f}(\vec{x}, t) + \nabla \cdot \boldsymbol{\sigma}(\vec{u}, p) \text{ in } \Omega$$

$$\boldsymbol{\sigma} \cdot \vec{n} = \vec{\tau}(\vec{x}, t) \text{ on } \Gamma_\tau$$

$$\vec{u} = \vec{u}_0(\vec{x}, t) \text{ on } \Gamma_u$$

$$\vec{q} \cdot \vec{n} = q_0(\vec{x}, t) \text{ on } \Gamma_q$$

$$p = p_0(\vec{x}, t) \text{ on } \Gamma_p$$

Dynamic Poroelasticity

Weak Formulation

Using trial functions $\vec{\psi}_{trial}^u$, ψ_{trial}^p , and $\vec{\psi}_{trial}^v$, and incorporating the Neumann boundary conditions, the weak form may be written as:

$$\int_{\Omega} \vec{\psi}_{trial}^u \cdot \left(\frac{\partial \vec{u}}{\partial t} \right) d\Omega = \int_{\Omega} \vec{\psi}_{trial}^u \cdot (\vec{v}) d\Omega$$

$$\int_{\Omega} \psi_{trial}^p \left(\frac{1}{M} \frac{\partial p}{\partial t} \right) d\Omega = \int_{\Omega} \psi_{trial}^p \left[\gamma(\vec{x}, t) - \alpha (\nabla \cdot \vec{u}) \right] + \nabla \psi_{trial}^p \cdot \vec{q}(p) d\Omega + \int_{\Gamma_q} \psi_{trial}^p (-q_0(\vec{x}, t)) d\Gamma$$

$$\int_{\Omega} \vec{\psi}_{trial}^v \cdot \left(\rho_b \frac{\partial \vec{v}}{\partial t} \right) d\Omega = \int_{\Omega} \vec{\psi}_{trial}^v \cdot \vec{f}(\vec{x}, t) + \nabla \vec{\psi}_{trial}^v : -\boldsymbol{\sigma}(\vec{u}, p_f) d\Omega + \int_{\Gamma_r} \vec{\psi}_{trial}^u \cdot \vec{r}(\vec{x}, t) d\Gamma.$$

Residual Functions

for Dynamic Isotropic Linear Poroelasticity

$$G^u(t, s) = \int_{\Omega} \vec{\psi}_{trial}^u \cdot \begin{pmatrix} \vec{v} \\ \underbrace{\vec{g}_0^u} \end{pmatrix} d\Omega$$

$$G^p(t, s) = \int_{\Omega} \psi_{trial}^p \left[\underbrace{\gamma(\vec{x}, t) - \alpha(\nabla \cdot \dot{\vec{u}})}_{\mathbf{g}_0^p} \right] + \nabla \psi_{trial}^p \cdot \underbrace{\vec{q}(p)}_{\vec{g}_1^p} d\Omega + \int_{\Gamma_q} \psi_{trial}^p \left(\underbrace{q_0(\vec{x}, t)}_{\mathbf{g}_0^p} \right) d\Gamma,$$

$$G^v(t, s) = \int_{\Omega} \vec{\psi}_{trial}^v \cdot \underbrace{\vec{f}(\vec{x}, t)}_{\vec{g}_0^v} + \nabla \vec{\psi}_{trial}^v : \underbrace{-\boldsymbol{\sigma}(\vec{u}, p)}_{\mathbf{g}_1^v} d\Omega + \int_{\Gamma_\tau} \vec{\psi}_{trial}^v \cdot \underbrace{\vec{\tau}(\vec{x}, t)}_{\vec{g}_0^v} d\Gamma,$$

Jacobian Functions

for Dynamic Isotropic Linear Poroelasticity

These are the pointwise functions associated with M_u , M_p , and M_v for computing the lumped LHS Jacobian. We premultiply the RHS residual function by the inverse of the lumped LHS Jacobian while s_{tshift} remains on the LHS with \dot{s} . As a result, we use LHS Jacobian pointwise functions, but set $s_{tshift} = 1$. The LHS Jacobians are:

$$M_u = J_F^{uu} = \frac{\partial F^u}{\partial u} + s_{tshift} \frac{\partial F^u}{\partial \dot{u}} = \int_{\Omega} \psi_{triali}^u \underbrace{s_{tshift} \delta_{ij}}_{J_{f0}^{uu}} \psi_{basisj}^u d\Omega$$

$$M_p = J_F^{pp} = \frac{\partial F^p}{\partial p} + t_{shift} \frac{\partial F^p}{\partial \dot{p}} = \int_{\Omega} \psi_{trial}^p \underbrace{\left(s_{tshift} \frac{1}{M} \right)}_{J_{f0}^{pp}} \psi_{basis}^p d\Omega$$

$$M_v = J_F^{vv} = \frac{\partial F^v}{\partial v} + t_{shift} \frac{\partial F^v}{\partial \dot{v}} = \int_{\Omega} \psi_{triali}^v \underbrace{\rho_b(\vec{x}) s_{tshift} \delta_{ij}}_{J_{f0}^{vv}} \psi_{basisj}^v d\Omega$$

Porothermoelastic Assumptions

Assumptions for Current Implementation

- Quasi-Static / Inertial terms ignored
- Nonlinear advective effects ignored
- Deformation assumed elastic

$$\vec{s} = [\vec{u}^T, p, T, \epsilon_v]^T$$

Conservation Equations

$$\nabla \cdot \boldsymbol{\sigma} + \vec{f} = \vec{0}$$

$$\frac{\dot{p}_f}{M} - \alpha\beta_b\dot{T} + \alpha\dot{\epsilon}_v + \nabla \cdot \vec{q} - \gamma = \vec{0}$$

$$\nabla \cdot \vec{u} - \epsilon_v = 0$$

$$(\rho c)_b \dot{T} + T_0 \beta_b \dot{\epsilon}_v^e + \nabla \cdot (\rho_f c_f \vec{q} T - \lambda_b \nabla T) - \dot{H} = 0$$

Pointwise Residuals

$$F^u(t, s, \dot{s}) = \int_{\Omega} \left[\vec{\psi}_{trial}^u \cdot \underbrace{\vec{f}(\vec{x}, t)}_{f_0^u} + \nabla \vec{\psi}_{trial}^u : \underbrace{(-\boldsymbol{\sigma}[\vec{u}, p])}_{f_1^u} \right] d\Omega + \int_{\Gamma^\tau} \left[\vec{\psi}_{trial}^u \cdot \underbrace{\vec{\tau}(\vec{x}, t)}_{f_1^u} \right] d\Gamma$$

$$F^p(t, s, \dot{s}) = \int_{\Omega} \left[\underbrace{\psi_{trial}^p \left(\frac{\partial \zeta(\epsilon_v, p)}{\partial t} + \alpha \beta_b \dot{T} - \gamma(\vec{x}, t) \right)}_{f_0^p} + \nabla \psi_{trial}^p \cdot \underbrace{-\vec{q}(p)}_{\vec{f}_1^p} \right] d\Omega + \int_{\Gamma^q} \psi_{trial}^p \underbrace{[\vec{n} \cdot \vec{q}(p)]}_{f_0^p} d\Gamma$$

$$F^T(t, s, \dot{s}) = \int_{\Omega} \left[\underbrace{\psi_{trial}^T ((\rho c)_b \dot{T})}_{f_0^T} - \nabla \psi_{trial}^T \cdot \underbrace{((\rho c)_f \vec{q}_D T - \lambda_b \nabla T + \dot{H})}_{\vec{f}_1^T} \right] d\Omega - \int_{\Gamma} \psi_{trial}^T \underbrace{[((\rho c)_f \vec{q}_D T \lambda_b \nabla T) \cdot \vec{n}]}_{f_0^T} d\Omega$$

$$F^\epsilon(t, s, \dot{s}) = \int_{\Omega} \psi_{trial}^\epsilon \cdot \underbrace{(\nabla \cdot \vec{u} - \epsilon_v)}_{f_0^\epsilon} d\Omega$$

Multiphase Flow

Black Oil Formulation

- Three phase flow (Water/Oil/Gas)
- Requires consideration of saturations, capillary pressures
- Biot modulus, M becomes a monster

Multiphase Flow

Black Oil Formulation

$$\vec{\zeta} = \boldsymbol{\alpha} \epsilon_v + \mathbf{N} \vec{p}$$

$$\begin{bmatrix} \zeta_o \\ \zeta_w \\ \zeta_g \end{bmatrix} = \begin{bmatrix} \alpha_o & 0 & 0 \\ 0 & \alpha_w & 0 \\ 0 & 0 & \alpha_g \end{bmatrix} \epsilon_v + \begin{bmatrix} N_{oo} & N_{ow} & N_{og} \\ N_{wo} & N_{ww} & N_{wg} \\ N_{go} & N_{gw} & N_{gg} \end{bmatrix} \begin{bmatrix} p_o \\ p_w \\ p_g \end{bmatrix}$$

Multiphase Flow

Black Oil Formulation

$$\frac{m_w}{\rho_w} = \underbrace{\left(\phi S_w c_w - \phi \frac{dS_w}{dp_{co}} + S_w \frac{\alpha - \phi}{K_s} S_w \right)}_{N_{ww}} p_w + \underbrace{\left(\phi \frac{dS_w}{dP_{co}} + S_w \frac{\alpha - \phi}{K_s} S_o \right)}_{N_{wo}} p_o \\ + \underbrace{\left(S_w \frac{\alpha - \phi}{K_s} S_g \right)}_{N_{wg}} p_g + \underbrace{(S_w \alpha)}_{\alpha_w} \epsilon_v$$

Multiphase Flow

Black Oil Formulation

$$\frac{m_o}{\rho_o} = \underbrace{\left(\phi S_o c_o + \phi \left(-\frac{dS_w}{dp_{co}} + \frac{dS_g}{dp_{cg}} \right) + S_o \frac{\alpha - \phi}{K_s} S_o \right)}_{N_{oo}} p_o + \underbrace{\left(\phi \frac{dS_w}{dp_{co}} + S_o \frac{\alpha - \phi}{K_s} S_w \right)}_{N_{ow}} p_w \\ + \underbrace{\left(\phi \left(-\frac{dS_g}{dp_{cg}} \right) + S_o \frac{\alpha - \phi}{K_s} S_g \right)}_{N_{og}} p_g + \underbrace{(S_o \alpha)}_{\alpha_o} \epsilon_v$$

Multiphase Flow

Black Oil Formulation

$$\frac{m_g}{\rho_g} = \underbrace{\left(-\phi \frac{dS_g}{dP_{cg}} + S_g \frac{\alpha - \phi}{K_s} S_o \right)}_{N_{go}} p_o + \underbrace{\left(S_g \frac{\alpha - \phi}{K_s} S_w \right)}_{N_{gw}} p_w + \underbrace{\left(\phi S_g c_g + S_g \frac{\alpha - \phi}{K_s} S_g + \phi \frac{dS_g}{dp_{cg}} \right)}_{N_{gg}} + \underbrace{(S_g \alpha)}_{\alpha_g} \epsilon_v$$

References I

-  Aagaard, B. T., M. G. Knepley, and C. A. Williams (2013). "A domain decomposition approach to implementing fault slip in finite-element models of quasi-static and dynamic crustal deformation". In: *Journal of Geophysical Research: Solid Earth* 118.6, pp. 3059–3079. ISSN: 21699356. DOI: [10.1002/jgrb.50217](https://doi.org/10.1002/jgrb.50217). arXiv: [1308.5846](https://arxiv.org/abs/1308.5846).
-  Ellsworth, William L. (2013). "Injection-induced earthquakes". In: *Science* 341.6142, pp. 142–149. ISSN: 10959203. DOI: [10.1126/science.1225942](https://doi.org/10.1126/science.1225942). URL: <http://www.clas.ufl.edu/users/prwaylen/GEO2200%20Readings/Readings/Fracking/Earthquakes%20and%20fracking.pdf>.
-  Goebel, T. H.W., M. Weingarten, X. Chen, J. Haffner, and E. E. Brodsky (2017). "The 2016 Mw5.1 Fairview, Oklahoma earthquakes: Evidence for long-range poroelastic triggering at >40 km from fluid disposal wells". In: *Earth and Planetary Science Letters* 472, pp. 50–61. ISSN: 0012821X. DOI: [10.1016/j.epsl.2017.05.011](https://doi.org/10.1016/j.epsl.2017.05.011). URL: <http://dx.doi.org/10.1016/j.epsl.2017.05.011>.
-  Biot, Maurice A. (1941). "General theory of three-dimensional consolidation". In: *Journal of Applied Physics* 12.2, pp. 155–164. ISSN: 00218979. DOI: [10.1063/1.1712886](https://doi.org/10.1063/1.1712886). arXiv: [/dx.doi.org/10.1063/1.1712886](http://dx.doi.org/10.1063/1.1712886) [<http://>].

Parameters

for Quasistatic Isotropic Linear Poroelasticity

Category	Symbol	Description
Unknowns	\vec{u}	Displacement field
	p	Pressure field (corresponds to pore fluid pressure)
	ϵ_v	Volumetric (trace) strain
Derived quantities	σ	Cauchy stress tensor
	ϵ	Cauchy strain tensor
	ζ	Variation of fluid content (variation of fluid vol. per unit vol. of PM), $\alpha\epsilon_v + \frac{p}{M}$
	ρ_b	Bulk density, $(1 - \phi)\rho_s + \phi\rho_f$
	\vec{q}	Darcy flux, $-\frac{\mathbf{k}}{\mu_f} \cdot (\nabla p - \vec{f}_f)$
	M	Biot Modulus, $\frac{K_f}{\phi} + \frac{K_s}{\alpha - \phi}$
Common constitutive parameters	ρ_f	Fluid density
	ρ_s	Solid (matrix) density
	ϕ	Porosity
	\mathbf{k}	Permeability
	μ_f	Fluid viscosity
	K_s	Solid grain bulk modulus
	K_f	Fluid bulk modulus
	K_d	Drained bulk modulus
	α	Biot coefficient, $1 - \frac{K_d}{K_s}$
Source terms	\vec{f}	Body force per unit volume, for example: $\rho_b \vec{g}$
	\vec{f}_f	Fluid body force, for example: $\rho_f \vec{g}$
	γ	Source density; rate of injected fluid per unit volume of the porous solid