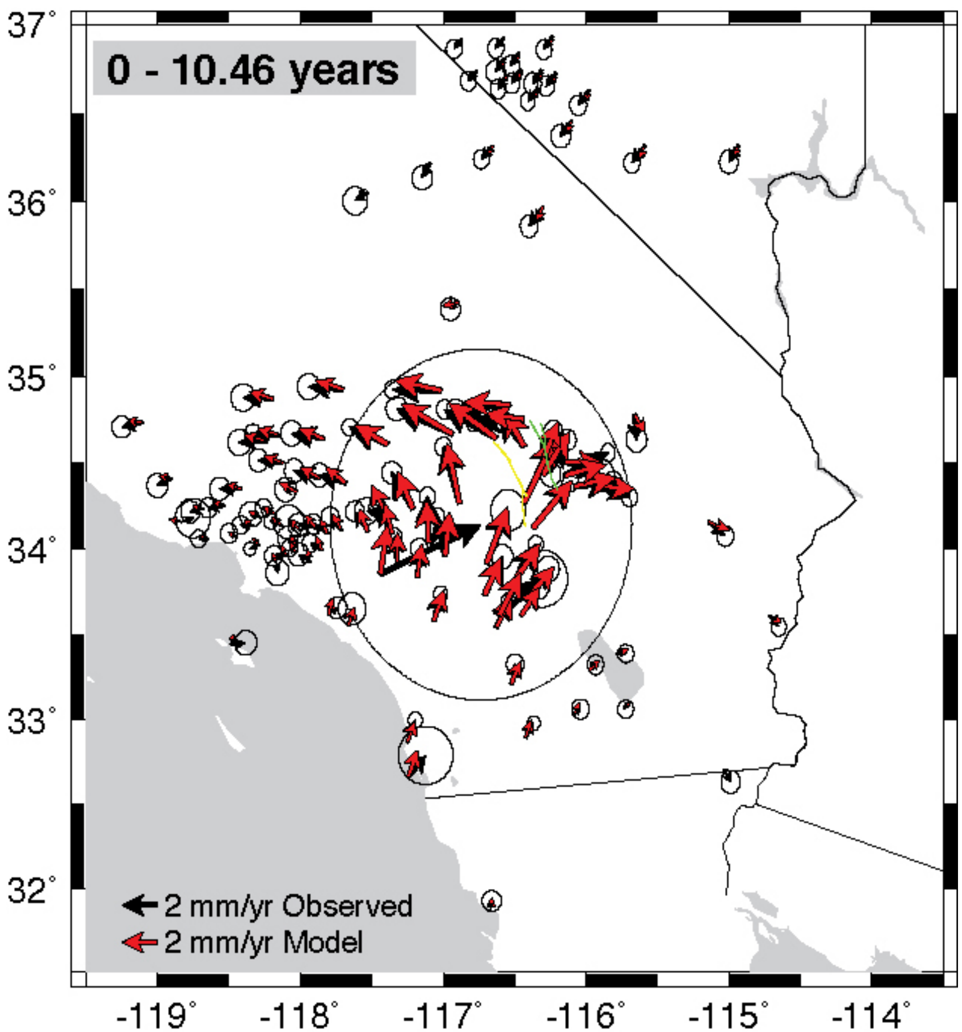
Post-Earthquake Relaxation Using a Spectral Element Method: 2.5D Case



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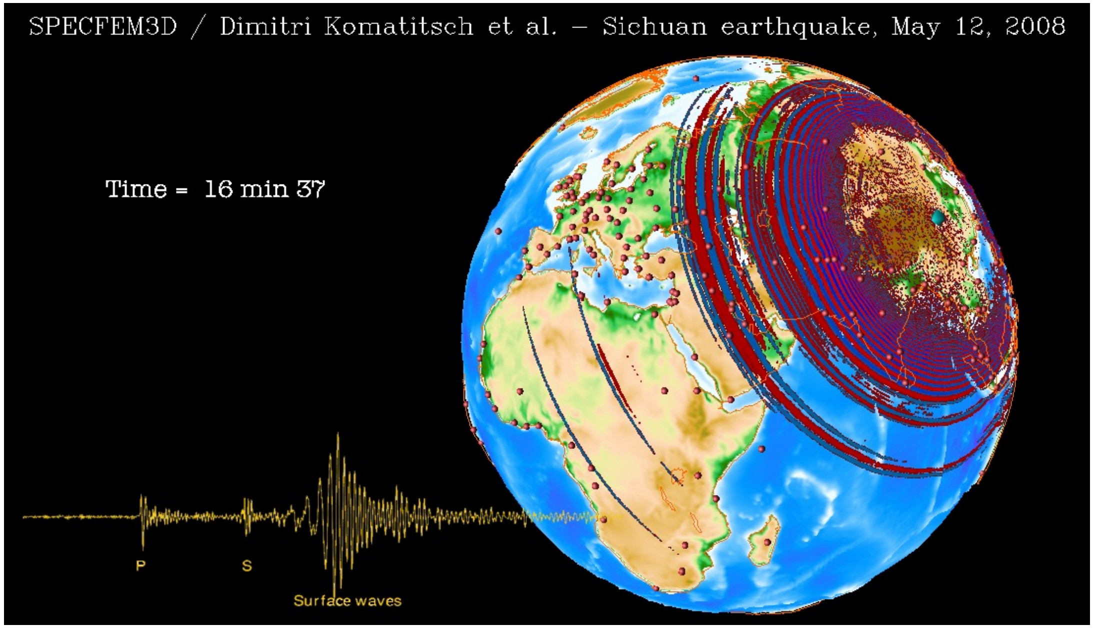
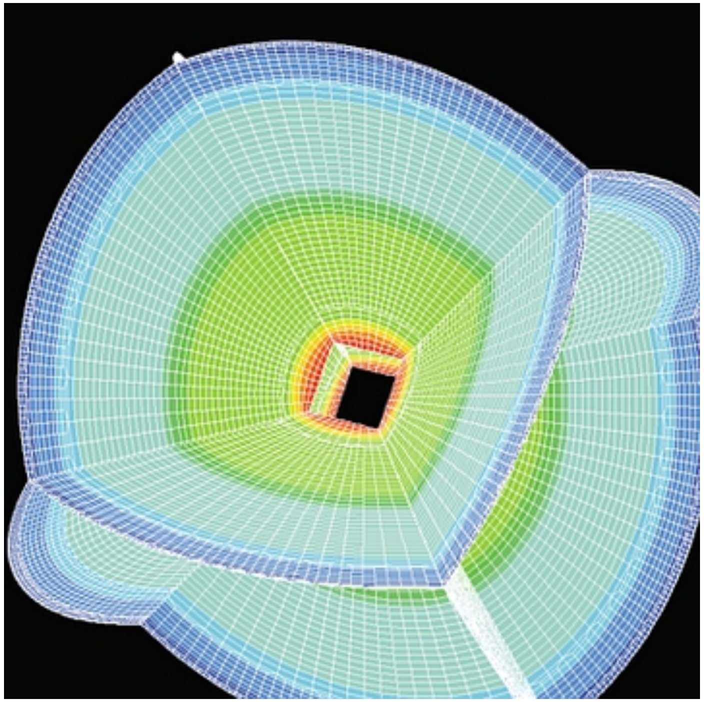
Seismic wave propagation

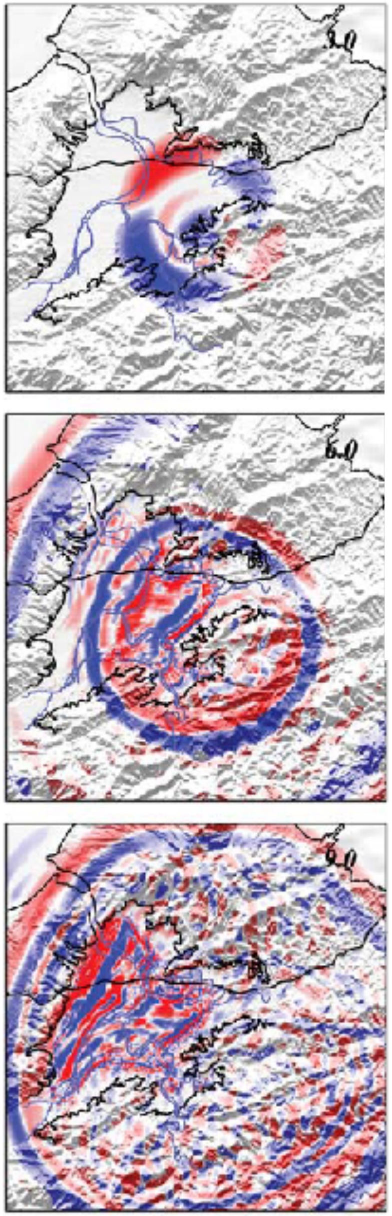


2008 Sichuan earthquake (Komatitsch et al. -- http://komatitsch.free.fr/)

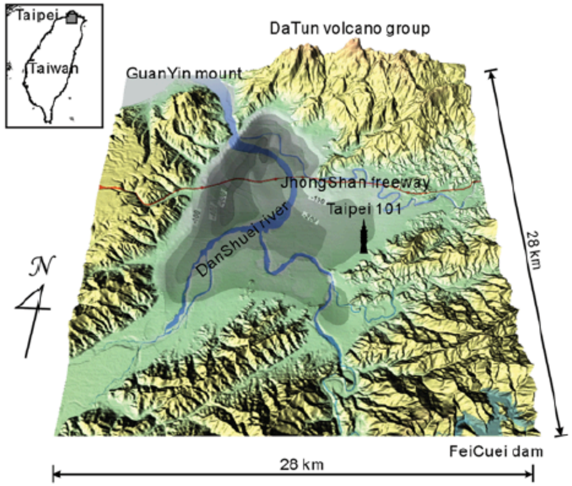
Komatitsch and Tromp (Linux Journal, 2001)



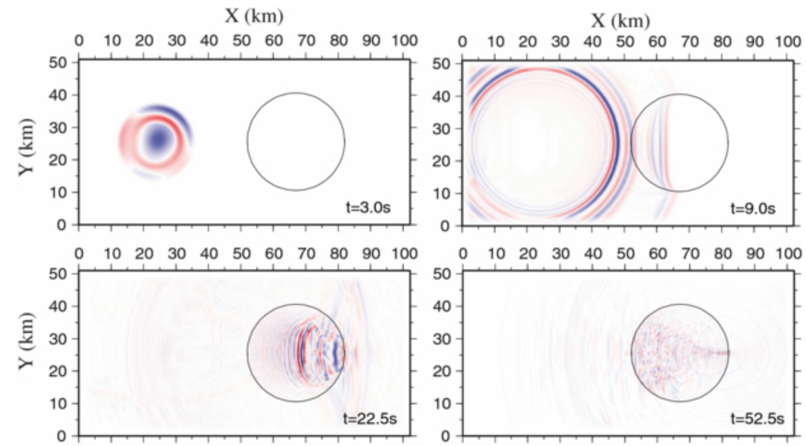
Seismic wave propagation



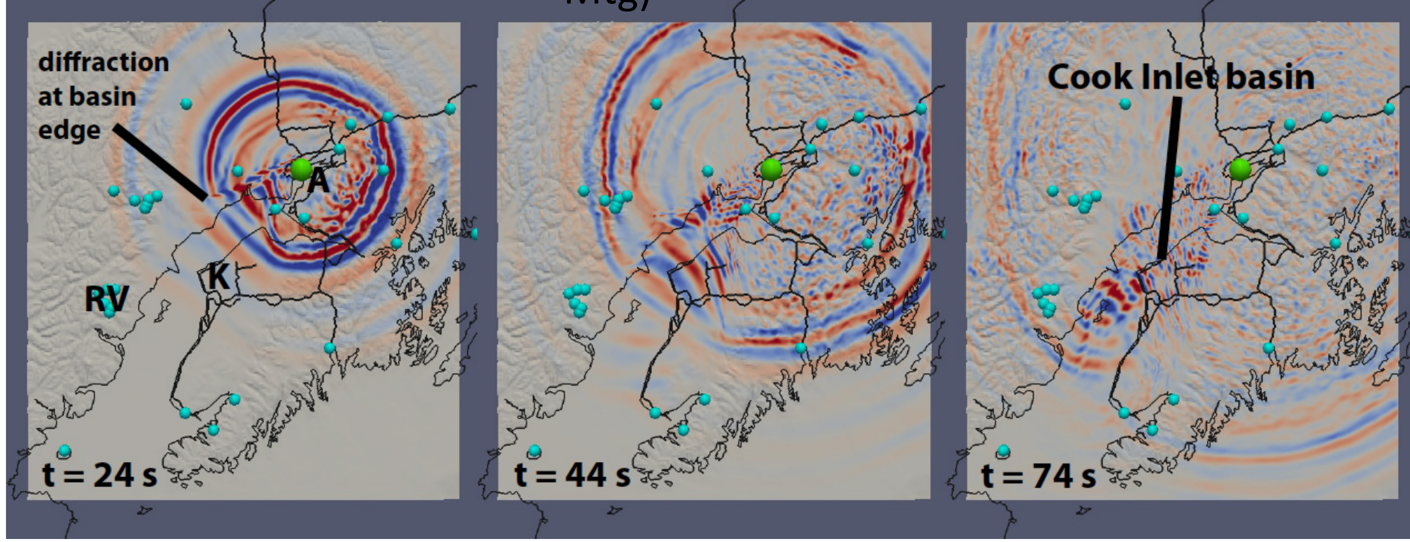
Taipai Basin
(Lee et al., 2008)



Sedimentary basin amplification
(Qin et al., 2012)



Cook Inlet Basin (Tape and Silwal, 2014 SSA Mtg)



Seismic wave propagation

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{r};s) + \mathbf{f}(\mathbf{r};s) = \rho(\mathbf{r})s^2\mathbf{u}(\mathbf{r};s)$$

$$\boldsymbol{\sigma}(\mathbf{r};s) = \mathbf{c}(\mathbf{r};s) : \nabla\mathbf{u}(\mathbf{r};s)$$

Quasi-static deformation

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{r};s) + \mathbf{f}(\mathbf{r};s) = \mathbf{0}$$

$$\boldsymbol{\sigma}(\mathbf{r};s) = \mathbf{c}(\mathbf{r};s) : \nabla\mathbf{u}(\mathbf{r};s)$$

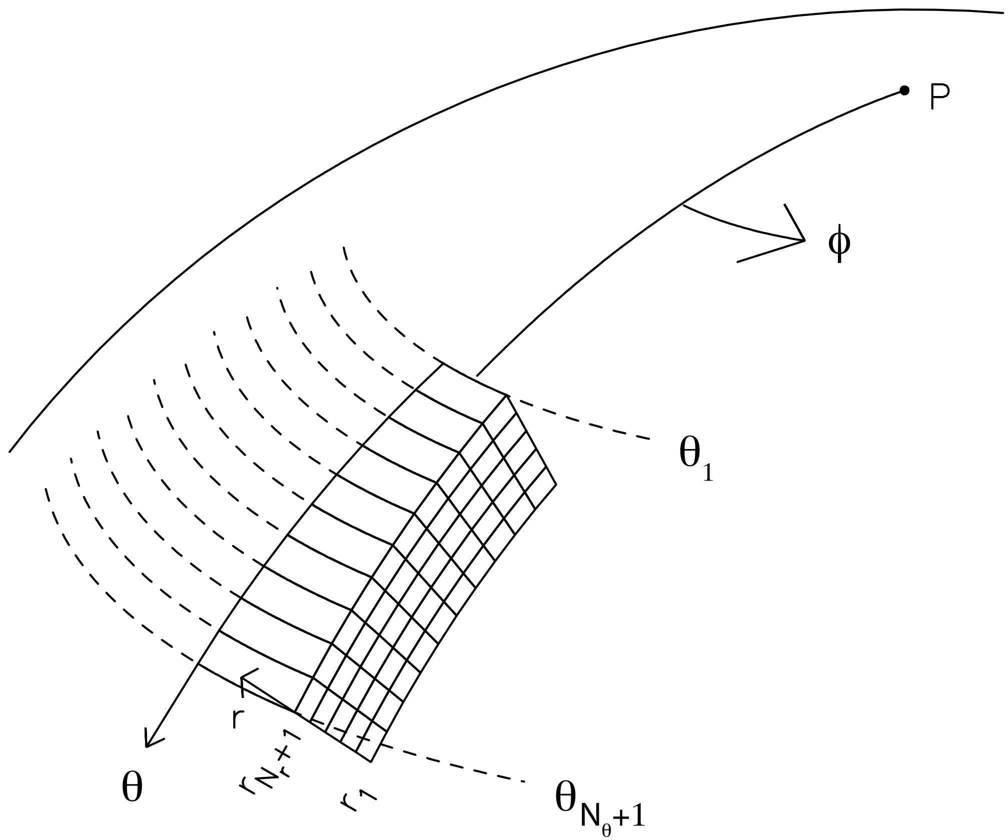
u = displacement; **σ** = stress tensor; **f** = source term
c = elastic tensor
r = position vector; s = Laplace transform variable

Outline

- SEM for quasi-static deformation in 2.5D
- Comparison of SEM with analytic solution
- Application to post-Landers and Hector Mine relaxation

Steps for implementing SEM

- Equations of quasi-static equilibrium → Strong form
  Weak form
- Model discretization → Mesh of elements
  → Transformation between physical and local elemental coordinates
  = Mapping
- Interpolation of functions on the elements → Lagrange polynomials
  Gauss-Lobatto-Legendre (GLL) points
- Integration over the element → GLL integration quadrature
  GLL points and weights
- ~~Mass matrix~~ → Quasi-static problem
  Stiffness matrix
- Assembly of global linear system



Geometry of model domain in 2.5D. Viscoelastic structure is assumed symmetric with respect to a symmetry pole P, i.e., it does not vary with ϕ, but may vary with θ and r.

Equations of quasi-static equilibrium

$$-\rho(\mathbf{r})\boldsymbol{\nabla}\Phi_1(r) + \boldsymbol{\nabla}\left[\rho(\mathbf{r})\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}\right] - \boldsymbol{\nabla}\cdot\left[\rho(\mathbf{r})\mathbf{u}(\mathbf{r};s)\right]\mathbf{g} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\mathbf{r};s) + \mathbf{f}(\mathbf{r};s) = \mathbf{0}$$

$$\boldsymbol{\sigma}(\mathbf{r};s) = \mathbf{c}(\mathbf{r};s) : \boldsymbol{\nabla}\mathbf{u}(\mathbf{r};s)$$

$$\nabla^2\Phi_1(\mathbf{r}) = -4\pi G\,\boldsymbol{\nabla}\cdot\left[\rho(\mathbf{r})\mathbf{u}(\mathbf{r};s)\right]$$

where ρ is density, Φ_1 is perturbed gravitational potential, c is the elastic tensor, and g is the reference gravitational acceleration vector

$$\mathbf{g} = -g_0(r)\hat{\mathbf{r}}$$

Ignore coupling of the elastic deformation field with Φ_1 →

$$\boldsymbol{\nabla}\left[\rho(\mathbf{r})\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}\right] - \boldsymbol{\nabla}\cdot\left[\rho(\mathbf{r})\mathbf{u}(\mathbf{r};s)\right]\mathbf{g} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\mathbf{r};s) + \mathbf{f}(\mathbf{r};s) = \mathbf{0}$$

r = position vector; s = Laplace transform variable

Equations of quasi-static equilibrium

$$\boldsymbol{\nabla}\left[\rho(\mathbf{r})\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}\right] - \boldsymbol{\nabla}\cdot\left[\rho(\mathbf{r})\mathbf{u}(\mathbf{r};s)\right]\mathbf{g} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\mathbf{r};s) + \mathbf{f}(\mathbf{r};s) = \mathbf{0}$$

For a reference structure that is in hydrostatic equilibrium

$$\nabla\rho\times\nabla g = 0$$

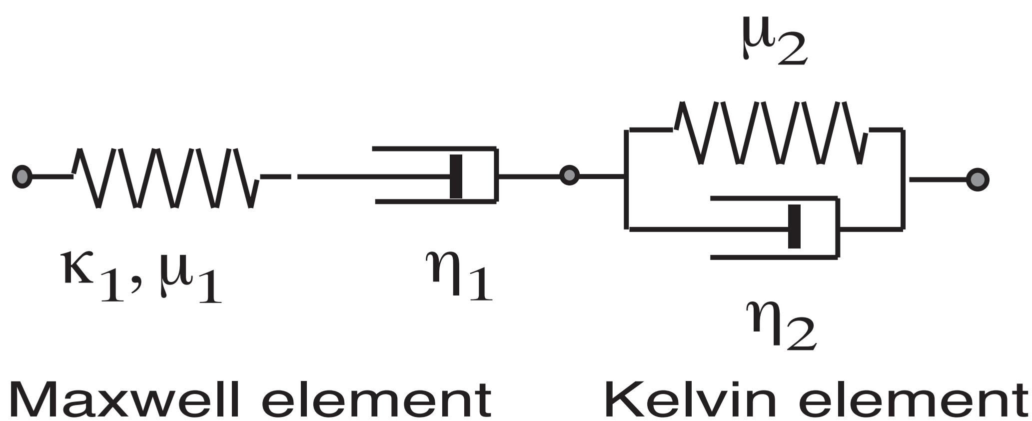
This enables us to re-write the equation of quasi-static equilibrium in a form which does not depend on the gradient of density:

$$\rho(\mathbf{r})\boldsymbol{\nabla}\left[\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}\right] - \rho(\mathbf{r})\left[\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{r};s)\right]\mathbf{g} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\mathbf{r};s) + \mathbf{f}(\mathbf{r};s) = \mathbf{0}$$

Viscoelasticity

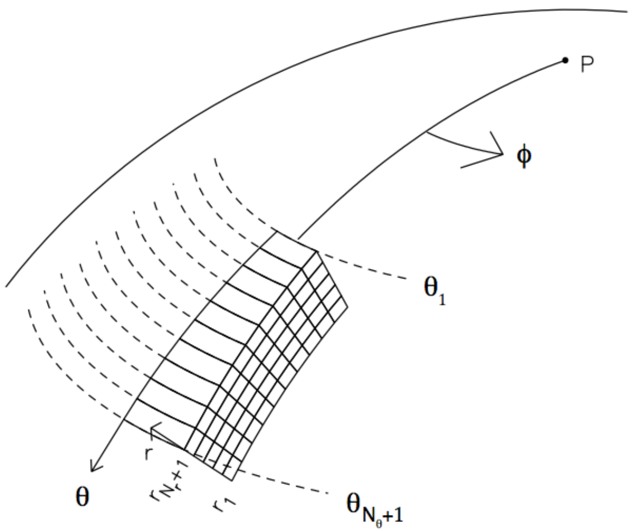
- Implemented in the Laplace transform domain

Burgers body: $$\mu(s) = \frac{\mu_1\, s\left(s + \frac{\mu_2}{\eta_2}\right)}{\left[\left(s + \frac{\mu_2}{\eta_2}\right)\left(s + \frac{\mu_1}{\eta_1}\right) + s\,\frac{\mu_1}{\eta_2}\right]}$$



Weak form of equation of quasi-static equilibrium

$$\rho(\mathbf{r})\boldsymbol{\nabla}\left[\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}\right]-\rho(\mathbf{r})\left[\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{r};s)\right]\mathbf{g}+\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\mathbf{r};s)+\mathbf{f}(\mathbf{r};s)=\mathbf{0}$$



Take the vector product with $w_j(r)$ and integrate over a volume V , the annulus swept out by an elemental area over the azimuth range from 0 to 2π about the pole of symmetry

$$\int_V \mathbf{w}_j(\mathbf{r})\cdot\left\{\rho(\mathbf{r})\boldsymbol{\nabla}\left[\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}\right]-\rho(\mathbf{r})\boldsymbol{\nabla}\cdot\left[\mathbf{u}(\mathbf{r};s)\right]\mathbf{g}\right\}\,d^3\mathbf{r}$$

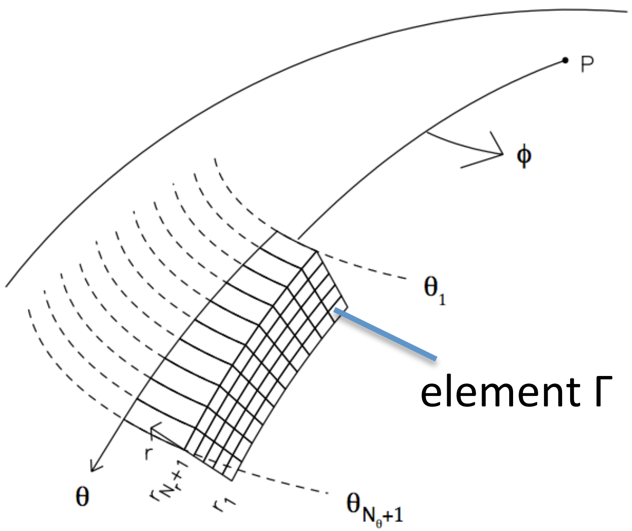
$$+\int_V \mathbf{w}_j(\mathbf{r})\cdot\left[\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\mathbf{r};s)\right]\,d^3\mathbf{r}=-\int_V \mathbf{w}_j(\mathbf{r})\cdot\mathbf{f}(\mathbf{r};s)\,d^3\mathbf{r}$$

Apply the divergence theorem to the $\nabla\cdot\sigma$ term to obtain the weak form:

$$\int_V \mathbf{w}_j(\mathbf{r})\cdot\left\{\rho(\mathbf{r})\boldsymbol{\nabla}\left[\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}\right]-\rho(\mathbf{r})\left[\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{r};s)\right]\mathbf{g}\right\}\,d^3\mathbf{r}+\int_{\partial V}\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}(\mathbf{r};s)\cdot\mathbf{w}_j(\mathbf{r})\,d^2\mathbf{r}$$

$$-\int_V \boldsymbol{\nabla}\mathbf{w}_j(\mathbf{r}):\boldsymbol{\sigma}(\mathbf{r};s)\,d^3\mathbf{r}=-\int_V \mathbf{w}_j(\mathbf{r})\cdot\mathbf{f}(\mathbf{r};s)\,d^3\mathbf{r}$$

Model discretization



A general mapping of local to global coordinates in element Γ is

$$r = \frac{r_k + r_{k+1}}{2} + z(r,\theta)\frac{\Delta r^\Gamma}{2} \qquad (-1 \leq z(r,\theta) \leq 1)$$

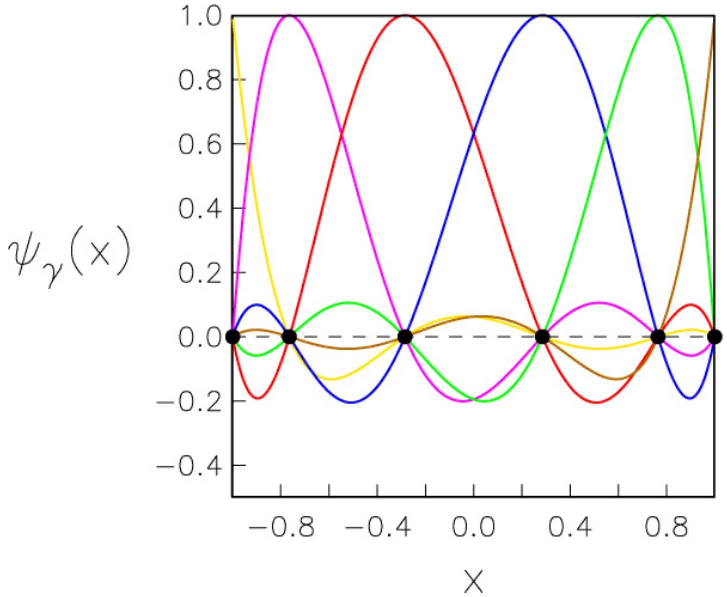
$$\theta = \frac{\theta_l + \theta_{l+1}}{2} + x(r,\theta)\frac{\Delta \theta^\Gamma}{2} \qquad (-1 \leq x(r,\theta) \leq 1)$$

In the Laplace transform domain, let the local 3D displacement field be expanded as

$$\mathbf{u}(r,\theta,\phi;s) = \sum_{m=-\infty}^{\infty}\sum_{\alpha=0}^{N}\sum_{\beta=0}^{N}\left[a^m_{\alpha\beta}(s)\psi_{\alpha\beta}(x,z)\hat{\boldsymbol{\theta}} + b^m_{\alpha\beta}(s)\psi_{\alpha\beta}(x,z)\hat{\boldsymbol{\phi}} + c^m_{\alpha\beta}(s)\psi_{\alpha\beta}(x,z)\hat{\mathbf{r}}\right] e^{im\phi}$$

- $\psi_{\alpha\beta}$ are 2D Gauss-Legendre-Lobatto basis functions
- m = azimuthal order number

Gauss-Legendre-Lobatto basis functions



Example with N=5

Interpolation points

$$x_0 = -1, \quad x_\gamma = \text{zeroes of } P'_N(x), \quad x_N = 1 \;\; (1 \le \gamma \le N-1)$$

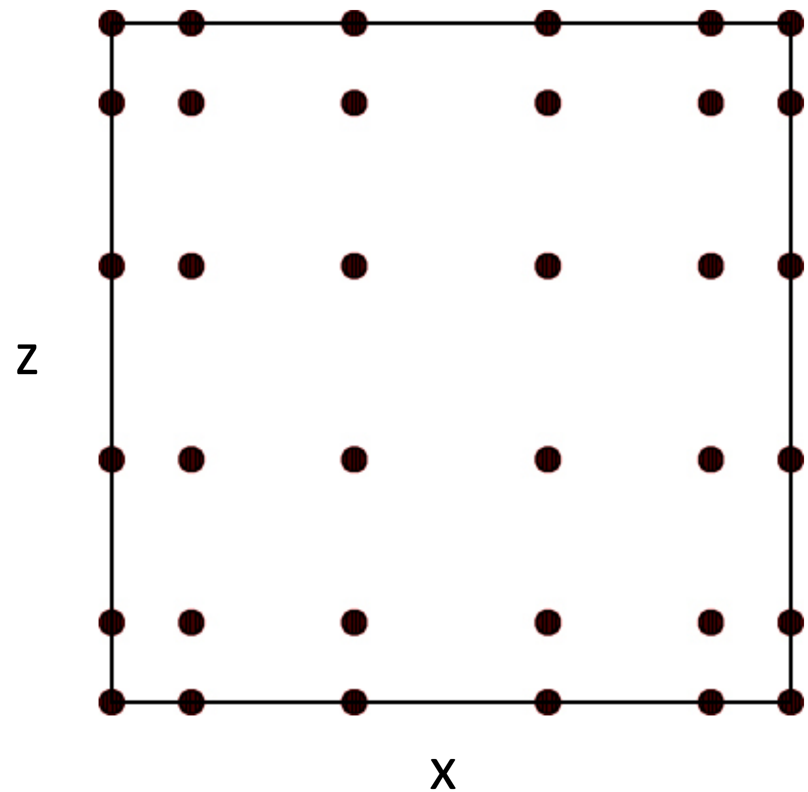
1D GLL basis functions

$$\psi_\gamma(x) = \frac{-1}{N(N+1)P_N(x_\gamma)} \frac{(1-x^2)P'_N(x)}{x - x_\gamma}$$

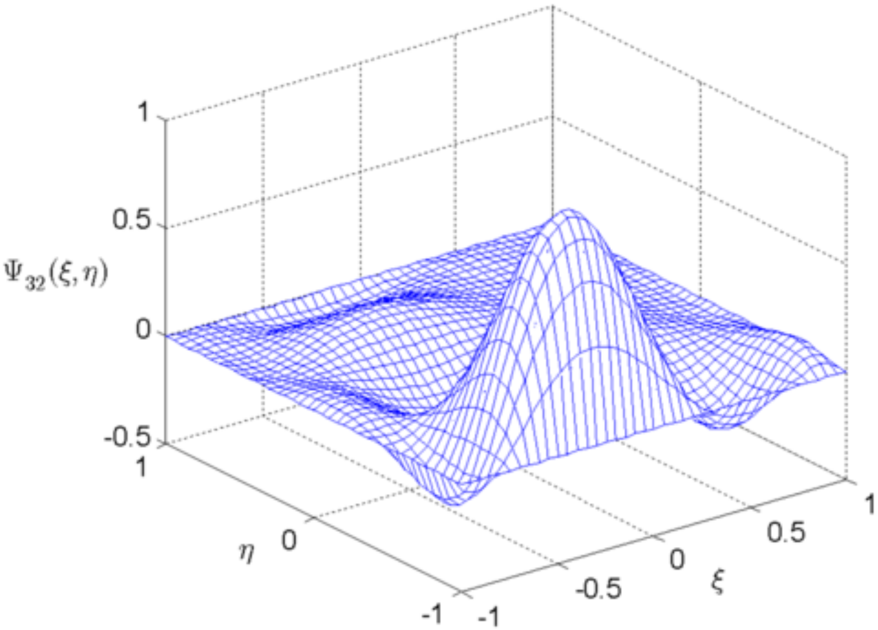
Gauss-Legendre-Lobatto basis functions

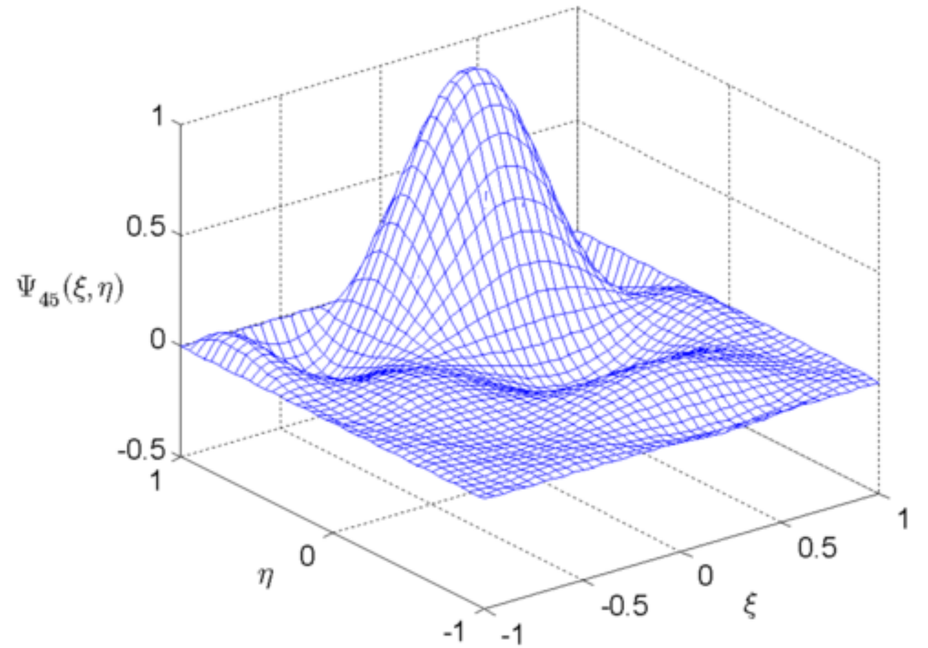
2D GLL basis functions

$$\psi_{\alpha\beta}(x,z) = \psi_\alpha(x)\,\psi_\beta(z)$$



Gauss-Legendre-Lobatto basis functions





Integration over the elements

Quadrature points

$$x_0 = -1, \quad x_\gamma = \text{zeroes of } P'_N(x), \quad x_N = 1 \quad (1 \le \gamma \le N-1)$$

2D Legendre-Gauss-Lobatto integration rule

$$\int_{-1}^{1}\int_{-1}^{1} f(x,z)\,dx\,dz \approx \sum_{\gamma}\sum_{\nu} w_\gamma w_\nu f(x_\gamma, x_\nu)$$

Weights

$$w_\gamma = \frac{2}{N(N+1)}\frac{1}{[P_N(x_\gamma)]^2} \quad (\gamma = 0, \cdots, N)$$

- Extremely high spatial accuracy -- 2D integration rule exact for polynomial functions up to degree 2N-1 in each dimension
- Quadrature points are identical to the interpolation points

Linear system of simultaneous equations

$$\int_V \mathbf{w}_j(\mathbf{r}) \cdot \{\rho(\mathbf{r})\nabla[\mathbf{u}(\mathbf{r};s)\cdot\mathbf{g}] - \rho(\mathbf{r})[\nabla\cdot\mathbf{u}(\mathbf{r};s)]\,\mathbf{g}\}\,d^3\mathbf{r} + \int_{\partial V} \hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}(\mathbf{r};s)\cdot\mathbf{w}_j(\mathbf{r})\,d^2\mathbf{r}$$

Weak form of eqns of static equilibrium

$$-\int_V \nabla\mathbf{w}_j(\mathbf{r}) : \boldsymbol{\sigma}(\mathbf{r};s)\,d^3\mathbf{r} = -\int_V \mathbf{w}_j(\mathbf{r})\cdot\mathbf{f}(\mathbf{r};s)\,d^3\mathbf{r}$$

$$\boldsymbol{\sigma}(\mathbf{r};s) = \mathbf{c}(\mathbf{r};s) : \nabla\mathbf{u}(\mathbf{r};s)$$

Stress-strain relation

Expansion of displacement field in terms of Gauss-Legendre-Lobatto basis functions

$$\mathbf{u}(r,\theta,\phi;s) = \sum_{m=-\infty}^{\infty}\sum_{\alpha=0}^{N}\sum_{\beta=0}^{N}\left[a^m_{\alpha\beta}(s)\psi_{\alpha\beta}(x,z)\hat{\boldsymbol{\theta}} + b^m_{\alpha\beta}(s)\psi_{\alpha\beta}(x,z)\hat{\boldsymbol{\phi}} + c^m_{\alpha\beta}(s)\psi_{\alpha\beta}(x,z)\hat{\mathbf{r}}\right] e^{im\phi}$$

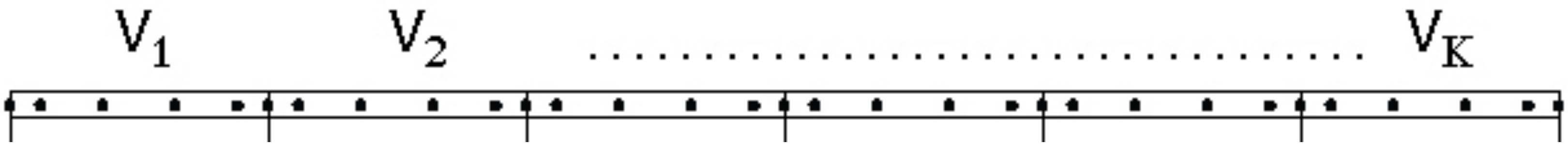
Test functions defined using GLL basis functions

$$\mathbf{w}_j(\mathbf{r}) = \frac{1}{2\pi}\psi_{\gamma\delta}(x(r,\theta),z(r,\theta))e^{-im\phi}\begin{cases}\hat{\boldsymbol{\theta}}, & \text{if } j=1\\ \hat{\boldsymbol{\phi}}, & \text{if } j=2\\ \hat{\mathbf{r}}, & \text{if } j=3\end{cases}$$

2D Gauss-Lobatto integration rule

$$\int_{-1}^{1}\int_{-1}^{1} f(x,z)\,dx\,dz \approx \sum_{\gamma}\sum_{\nu} w_\gamma w_\nu f(x_\gamma,x_\nu)$$

Assembly of global linear system



Weak form of eqns of static equilibrium

$$\int_V g(u(x))w_j(x)\,dx = -\int_V f(x)w_j(x)\,dx$$

Domain V divided into K non-overlapping subdomains V_k

$$\sum_{k=1}^{K}\left[\int_{V_k} g(u(x))w_j(x)\,dx\right] = -\sum_{k=1}^{K}\int_{V_k} f(x)w_j(x)\,dx$$

Global interpolant

$$u(x) = \sum_{k=1}^{K}\sum_{j=1}^{N} u_j^k(x)\,\phi_j^k(x)$$

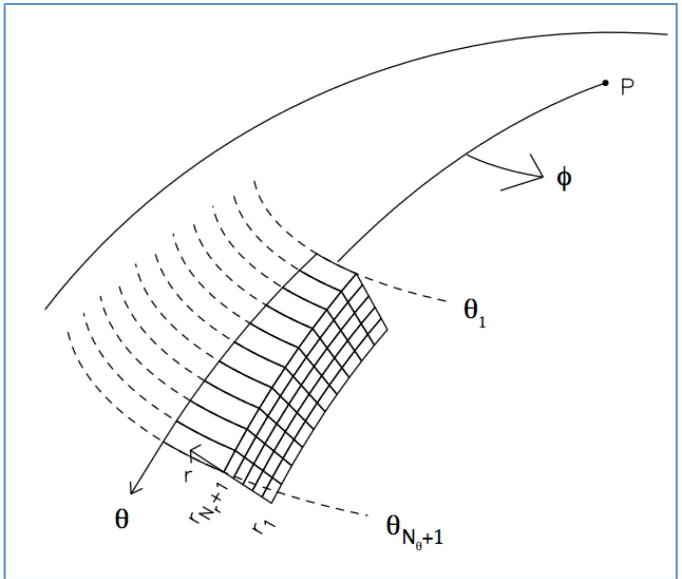
$$\phi_j^k = w_j(x) \text{ in the } k\text{th element and } 0 \text{ outside the } k\text{th element}$$

Linear system of simultaneous equations

$$\mathbf{KU} = \mathbf{F}$$

(Stiffness matrix) (Displacement coefficients) = Source vector

The vector of unknowns U contains the $3(N_{Nr}+1)(N_{N\theta}+1)$ expansion coefficients of the displacement components $(a^m_{\alpha\beta}, b^m_{\alpha\beta}, \text{and } c^m_{\alpha\beta})$, three for each of the $(N_{Nr}+1)(N_{N\theta}+1)$ nodes of the global grid



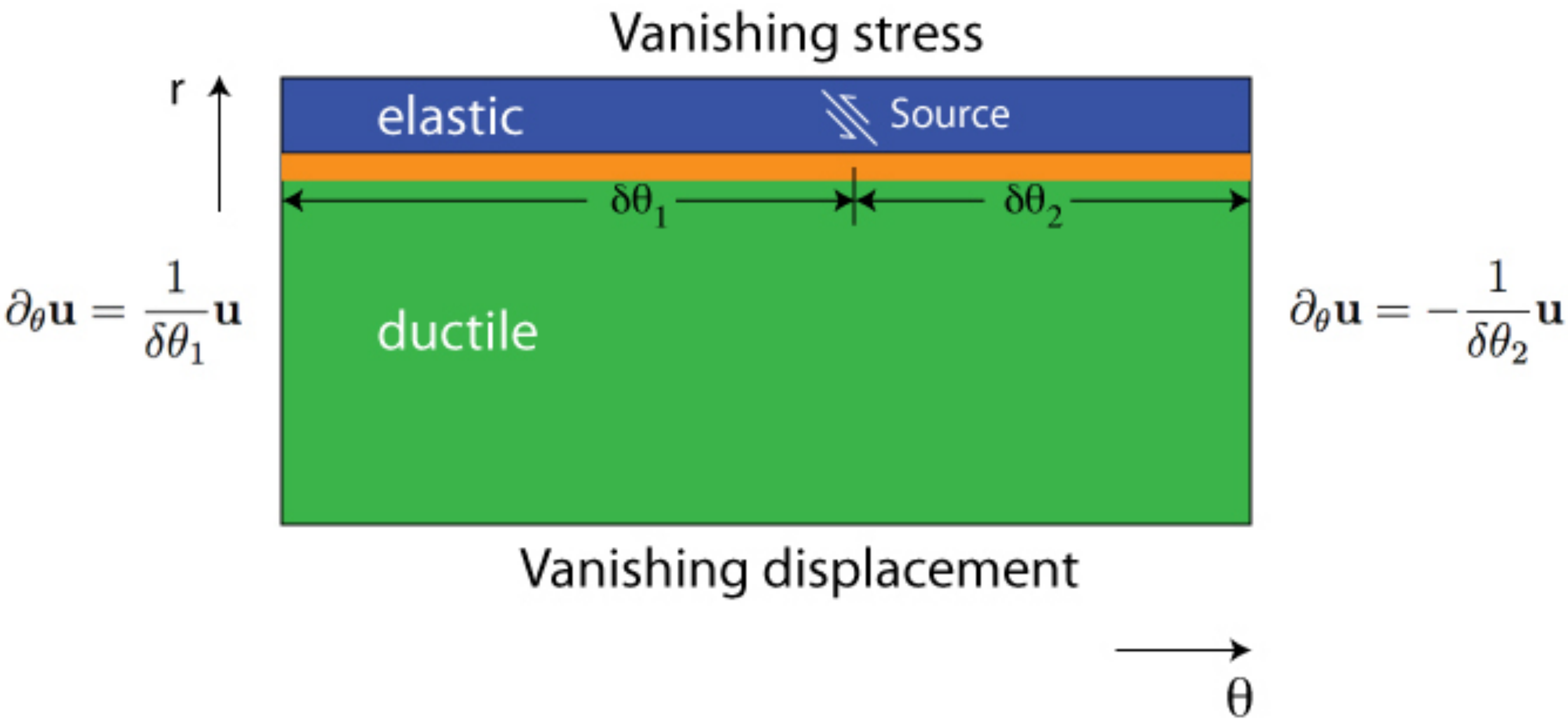
At the 2D GLL interpolation points

$$\psi_{\alpha\beta}(x_\gamma, x_\nu) = \delta_{\alpha\gamma}\delta_{\beta\nu}$$

Displacement field at points $(r_{\gamma\nu},\theta_{\gamma\nu})$ and arbitrary ϕ is

$$\mathbf{u}(r_{\gamma\nu},\theta_{\gamma\nu},\phi;s) = \sum_{m=-\infty}^{\infty} \left[a^m_{\gamma\nu}(s)\hat{\boldsymbol{\theta}} + b^m_{\gamma\nu}(s)\hat{\boldsymbol{\phi}} + c^m_{\gamma\nu}(s)\hat{\mathbf{r}}\right] e^{im\phi}$$

Boundary Conditions



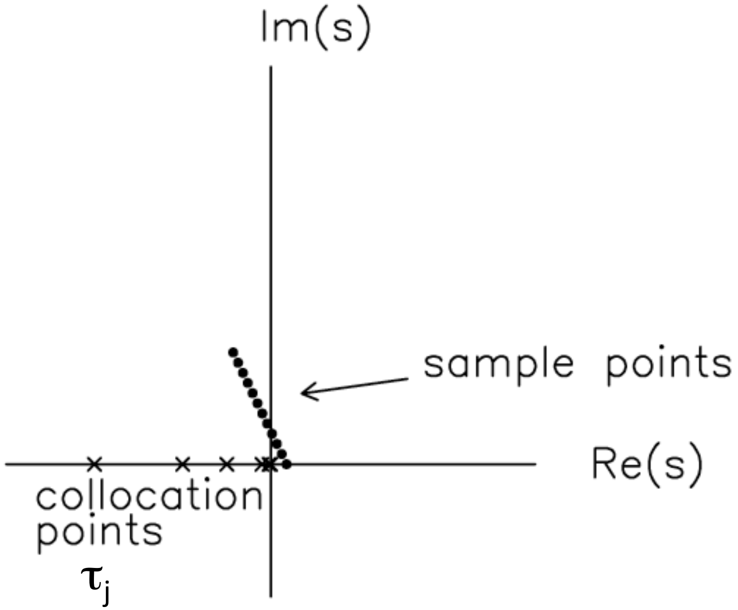
Time domain results

Laplace transform domain

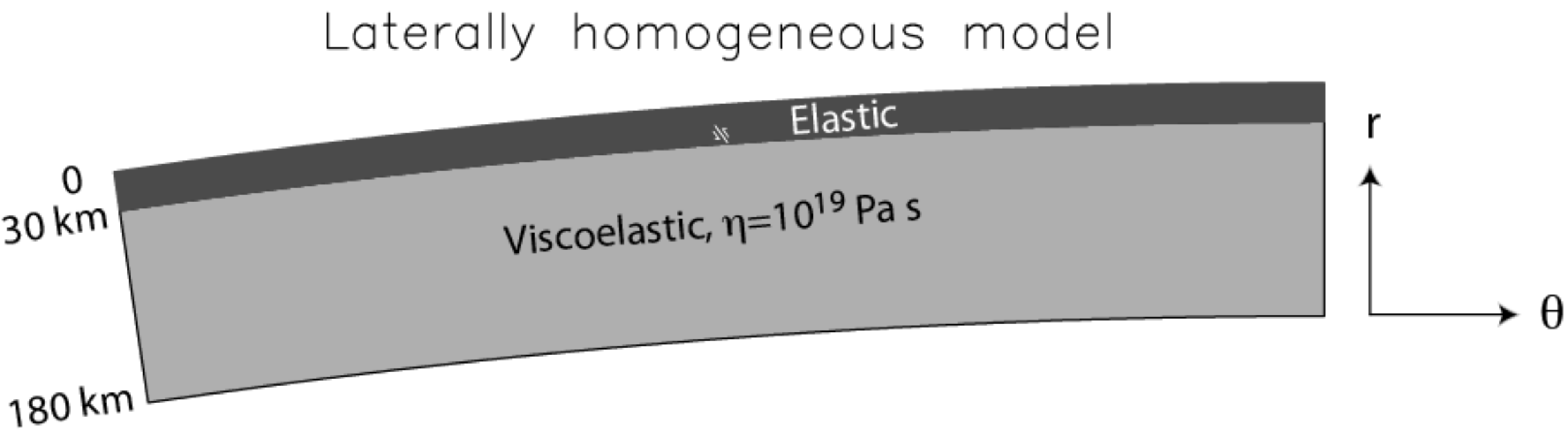
$$\tilde{u}(s) = \frac{B_0}{s} + \sum_{j=1}^{7} \frac{B_j}{s(s+\tau_j)}$$

Time domain

$$u(t) = B_0\, H(t) + \sum_{j=1}^{7} B_j \tau_j^{-1} \left(1 - e^{-\tau_j t}\right)$$

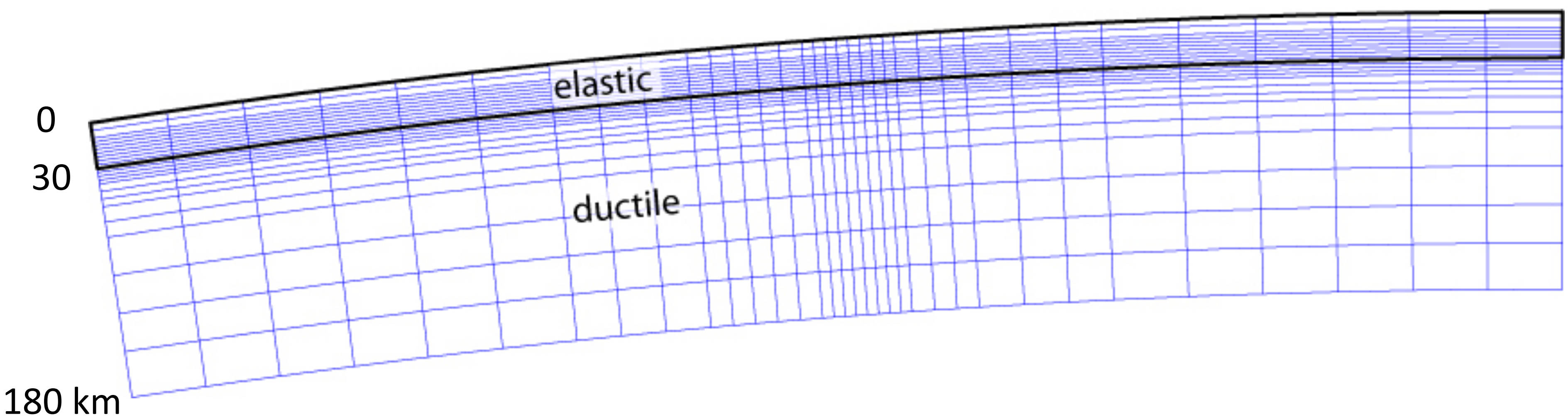


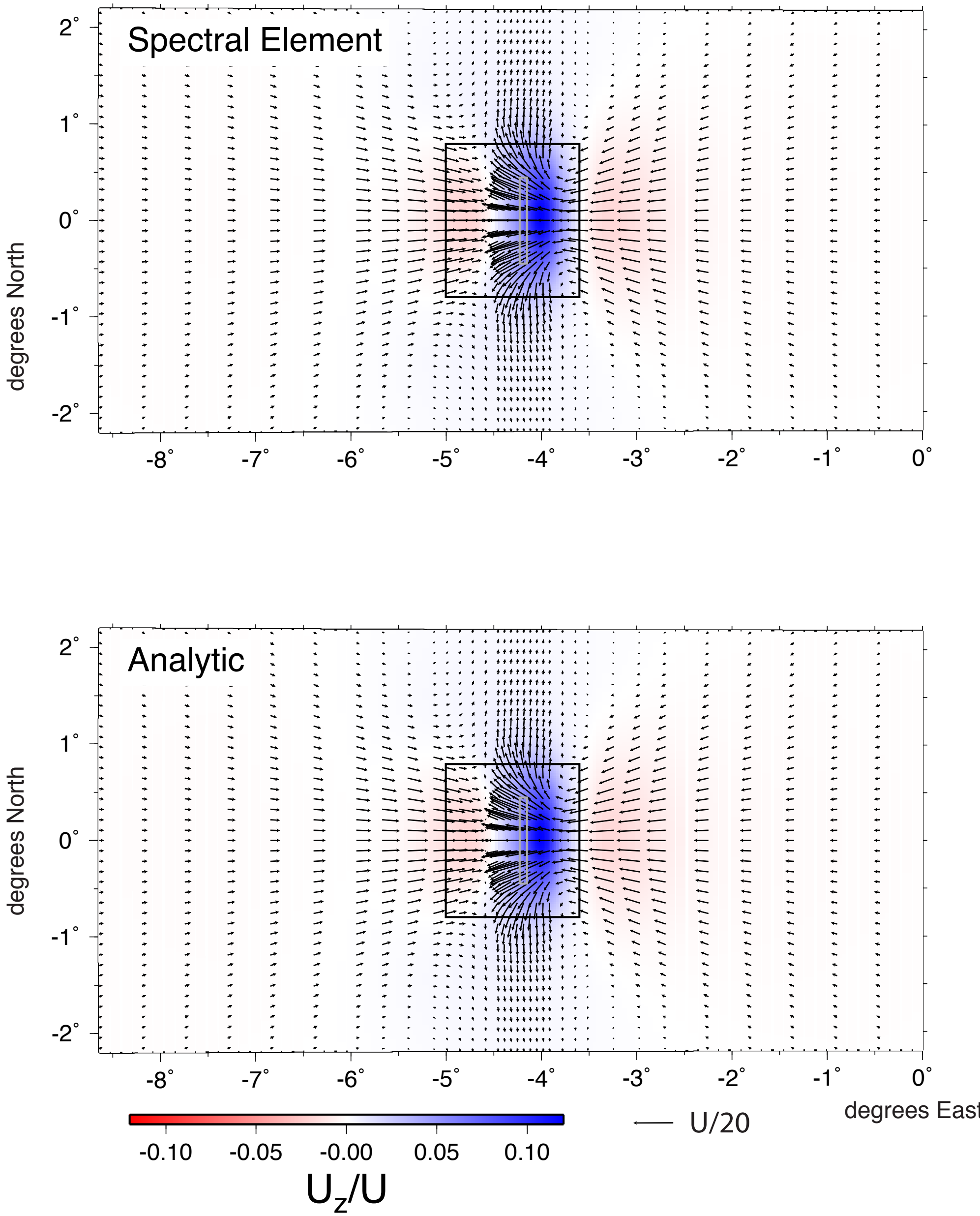
Comparison of Spectral Element Method with Analytic Solution



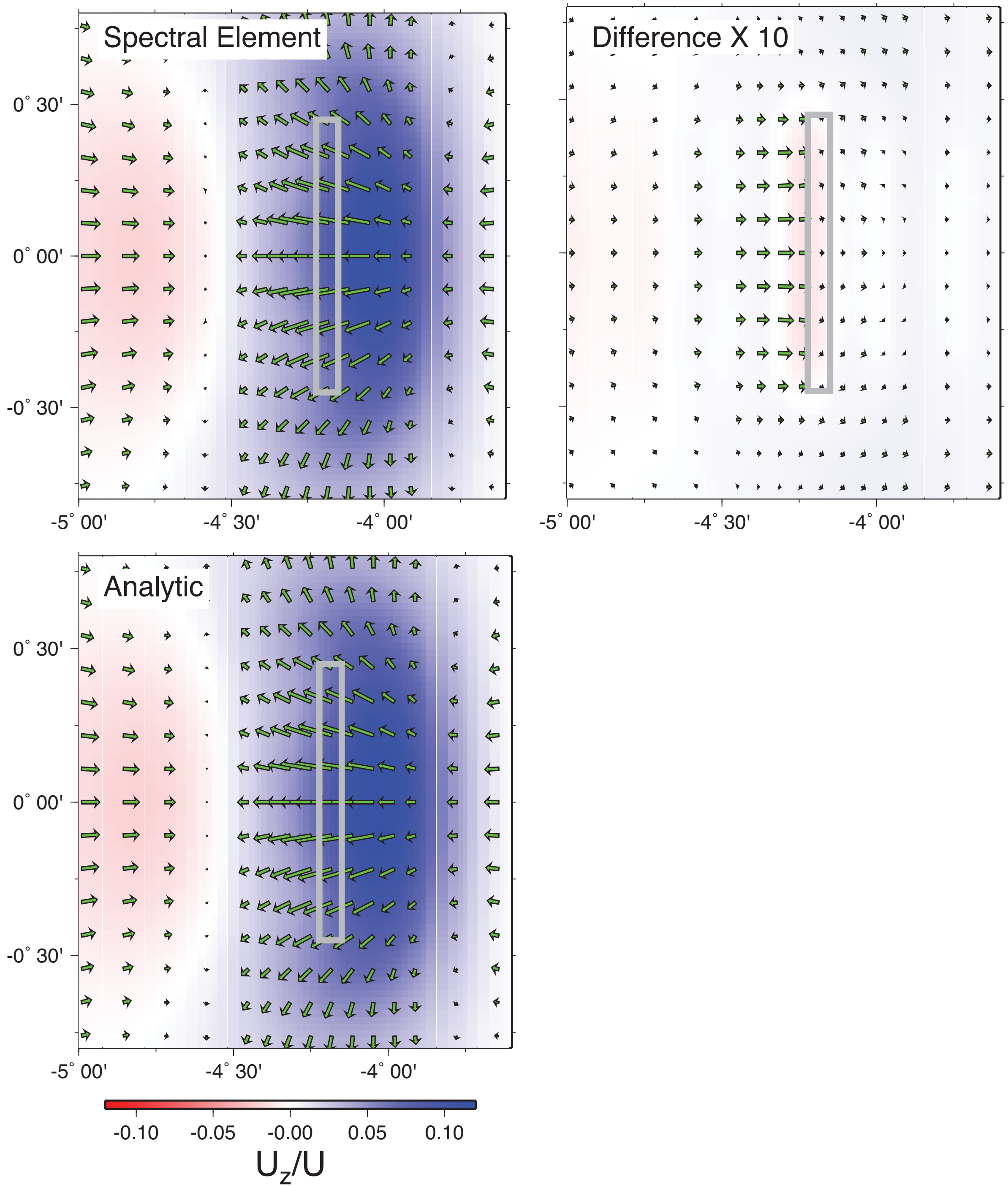
Details

- Uniform elastic parameters throughout
  $\kappa_0 = \lambda_0 + (2/3)\mu_0 = 50\ \text{GPa},\ \mu_0 = 30\ \text{GPa}$
- Uniform viscoelastic between 30 km and 180 km
- 34 x 27 cells, each with 36 GLL quadrature points
- Max azimuthal order number = 1353 in increments of 33 → repeating sources spaced 1200 km apart



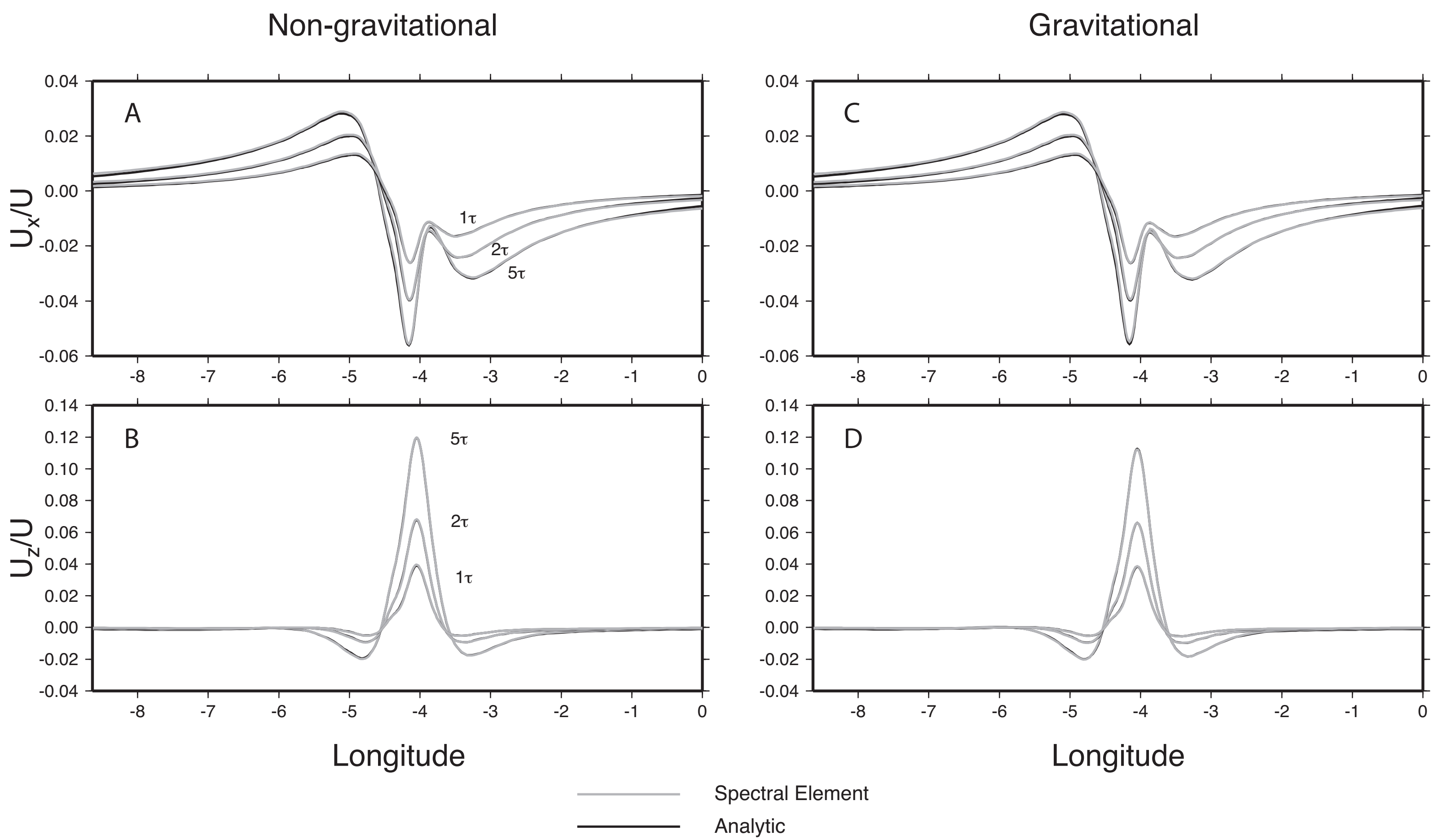


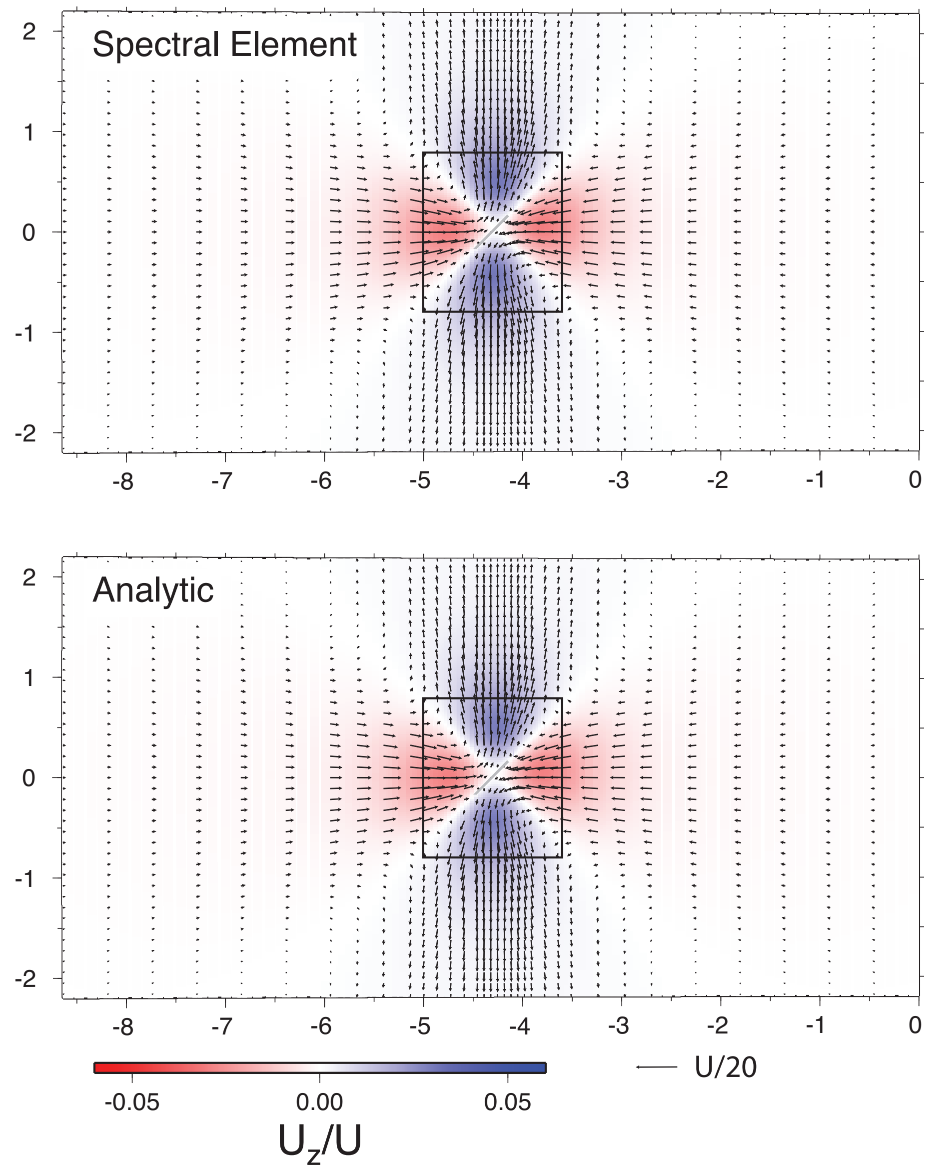
Cumulative displacements
up to 5τ for
post-thrusting relaxation



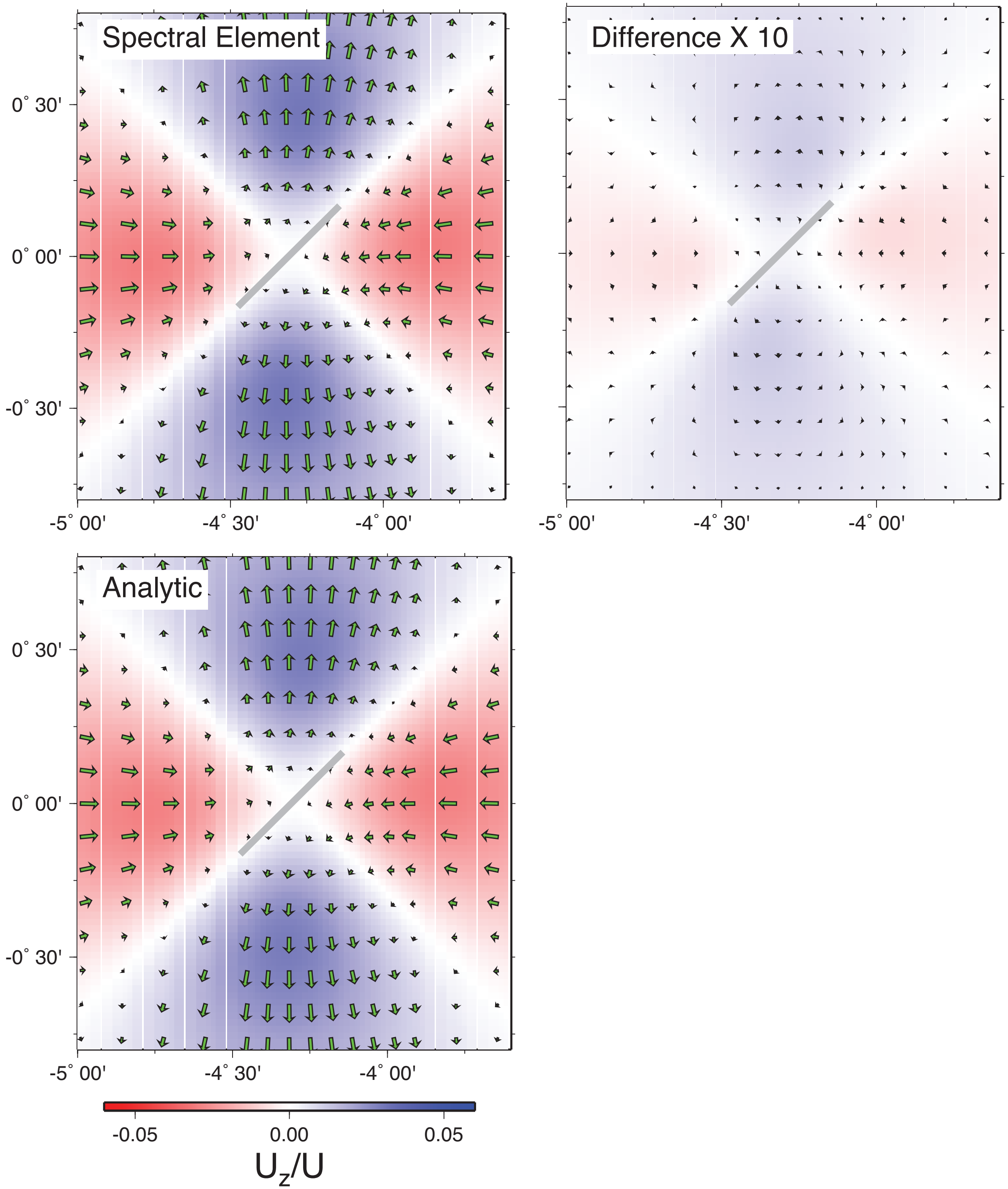
Cumulative displacements up to 5τ for post-thrusting relaxation

Cumulative displacements along the equator for post-thrusting relaxation



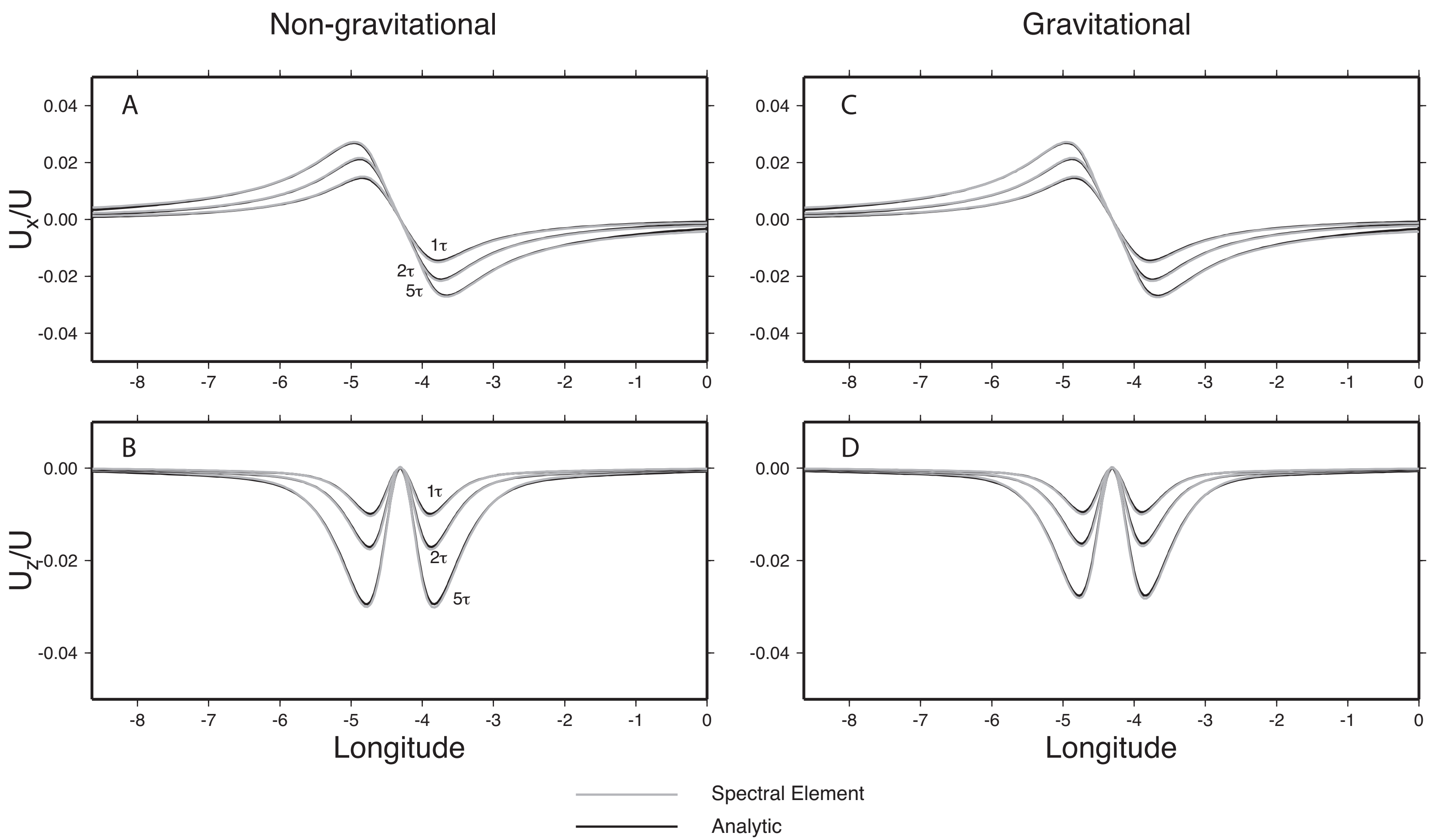


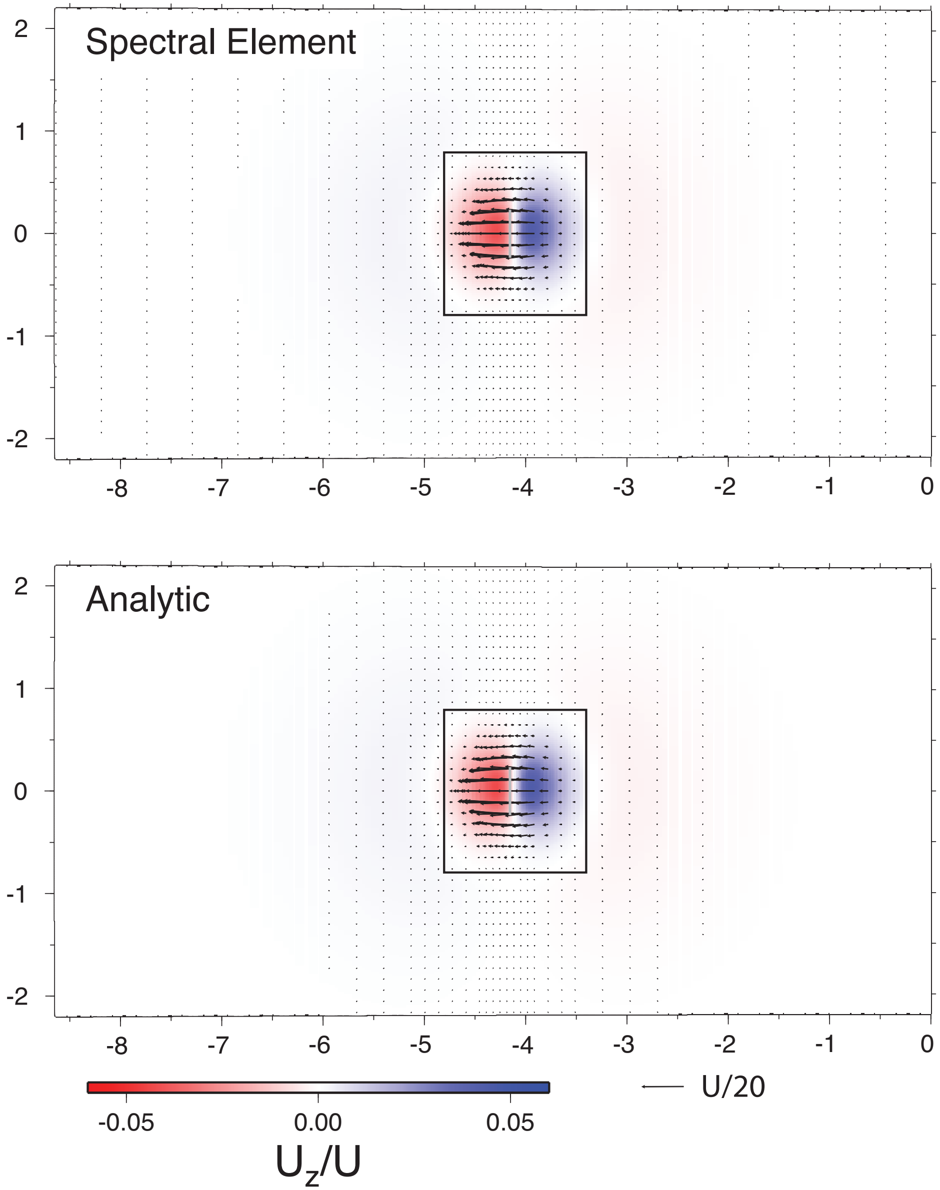
Cumulative displacements up to 5τ for post-strike-slip relaxation



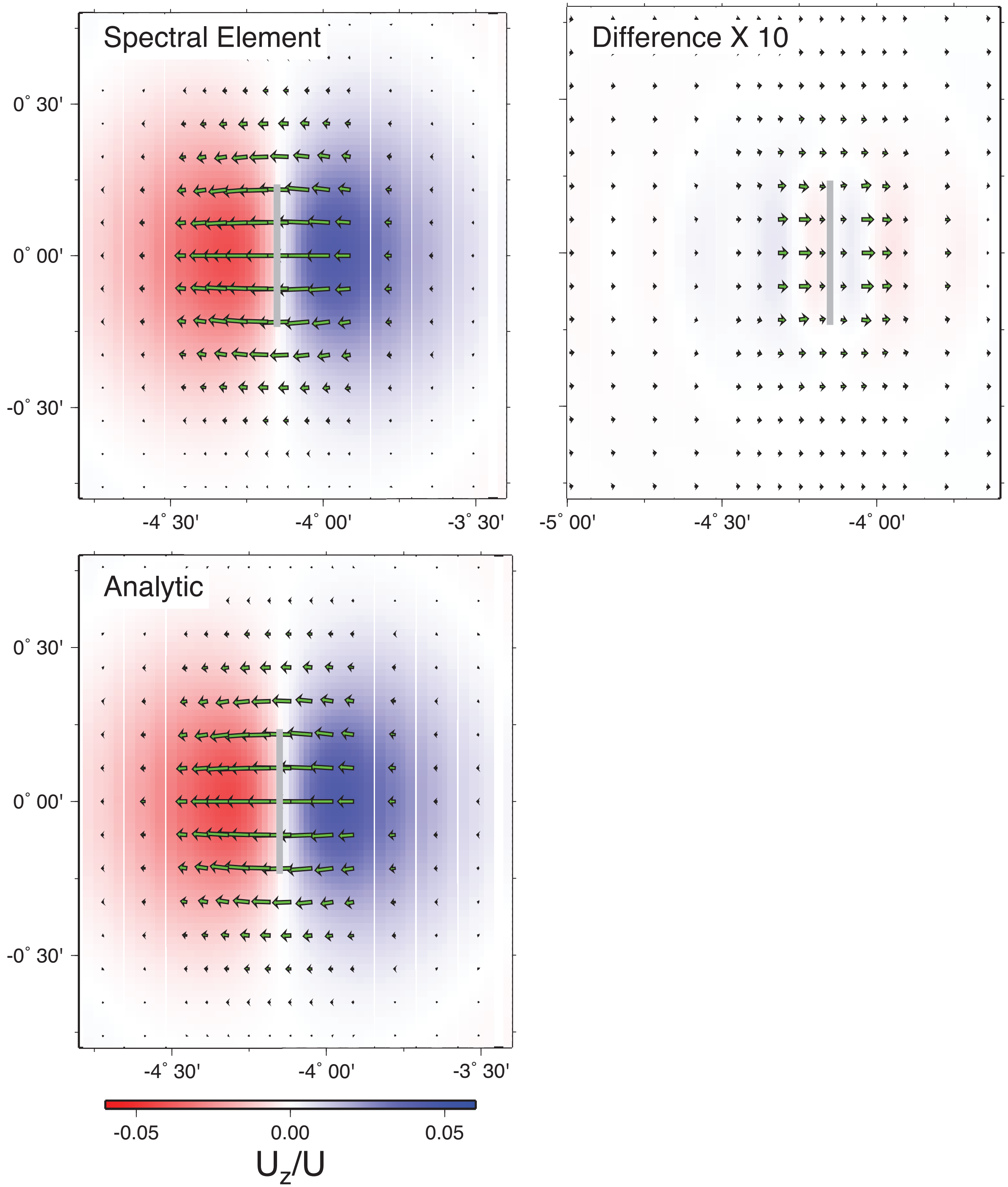
Cumulative displacements up to 5τ for post-strike-slip relaxation

Cumulative displacements along the equator for post-strike-slip relaxation



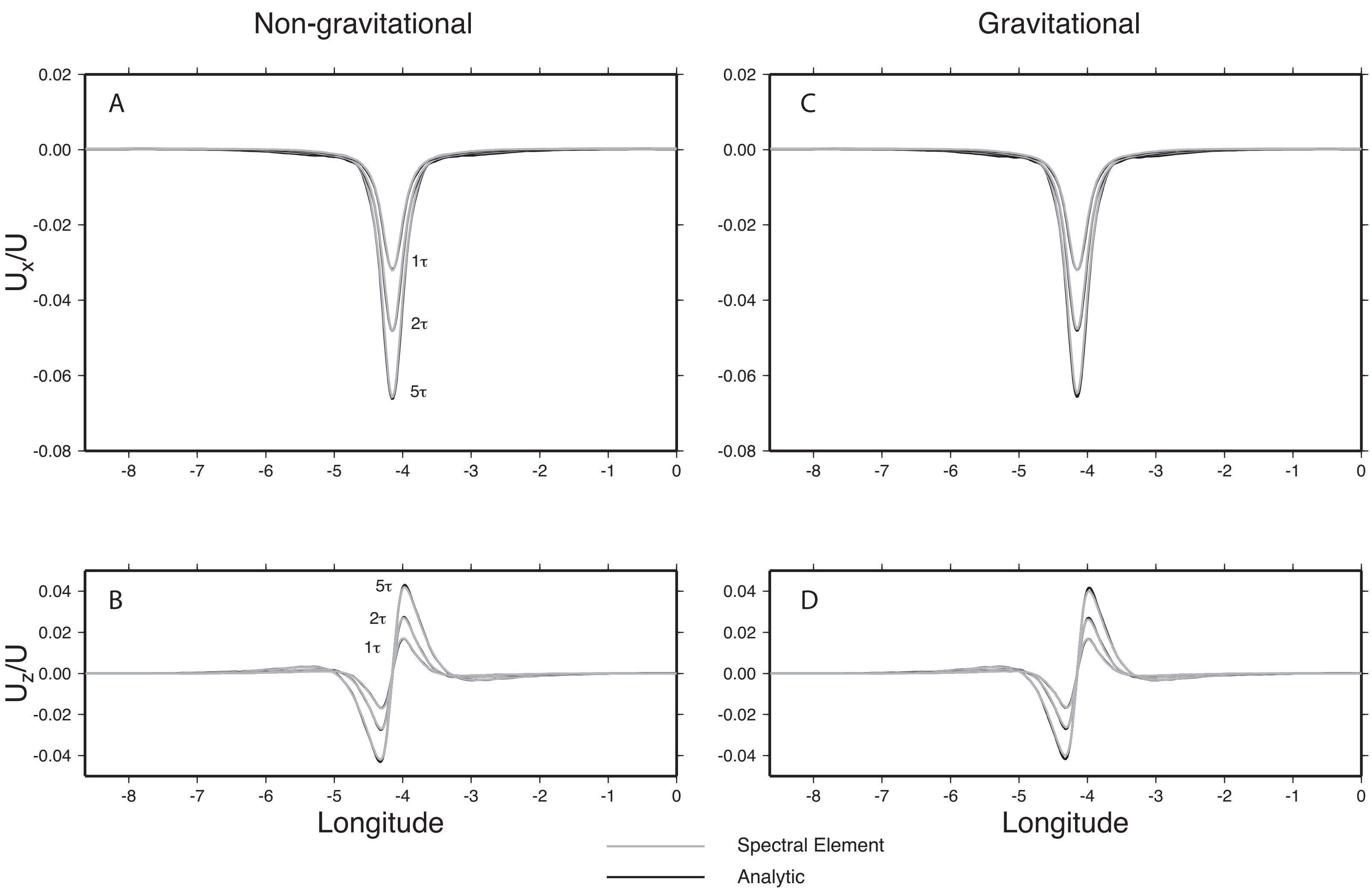


Cumulative displacements up to 5τ for post-dip-slip relaxation

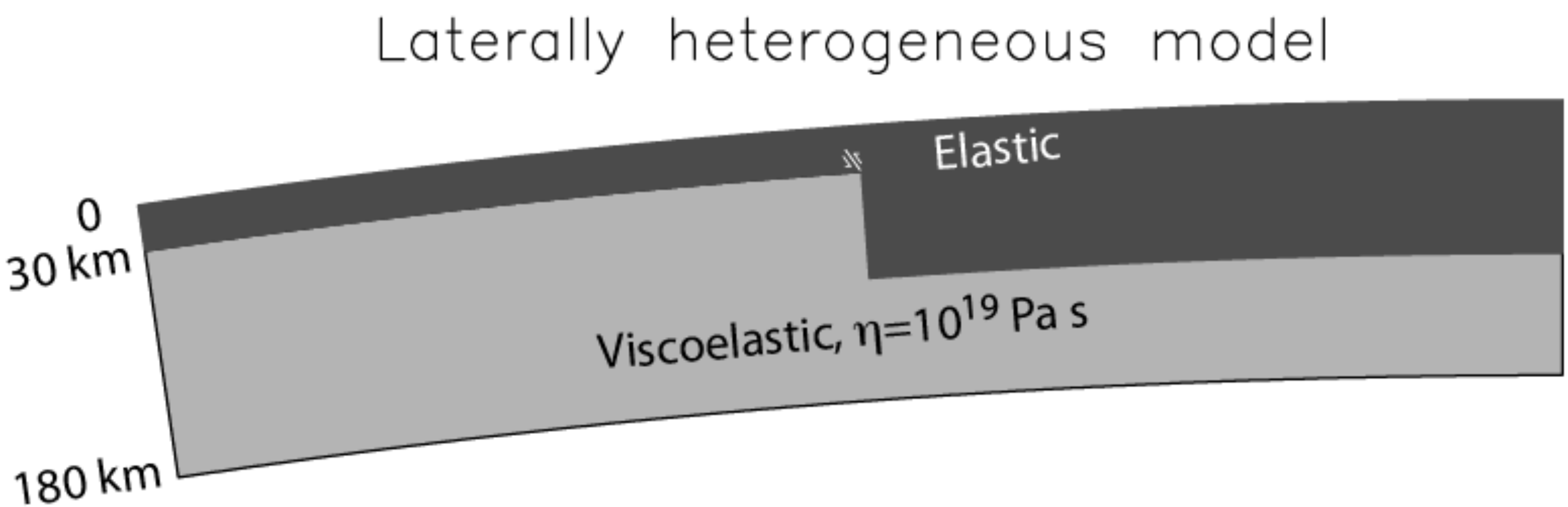


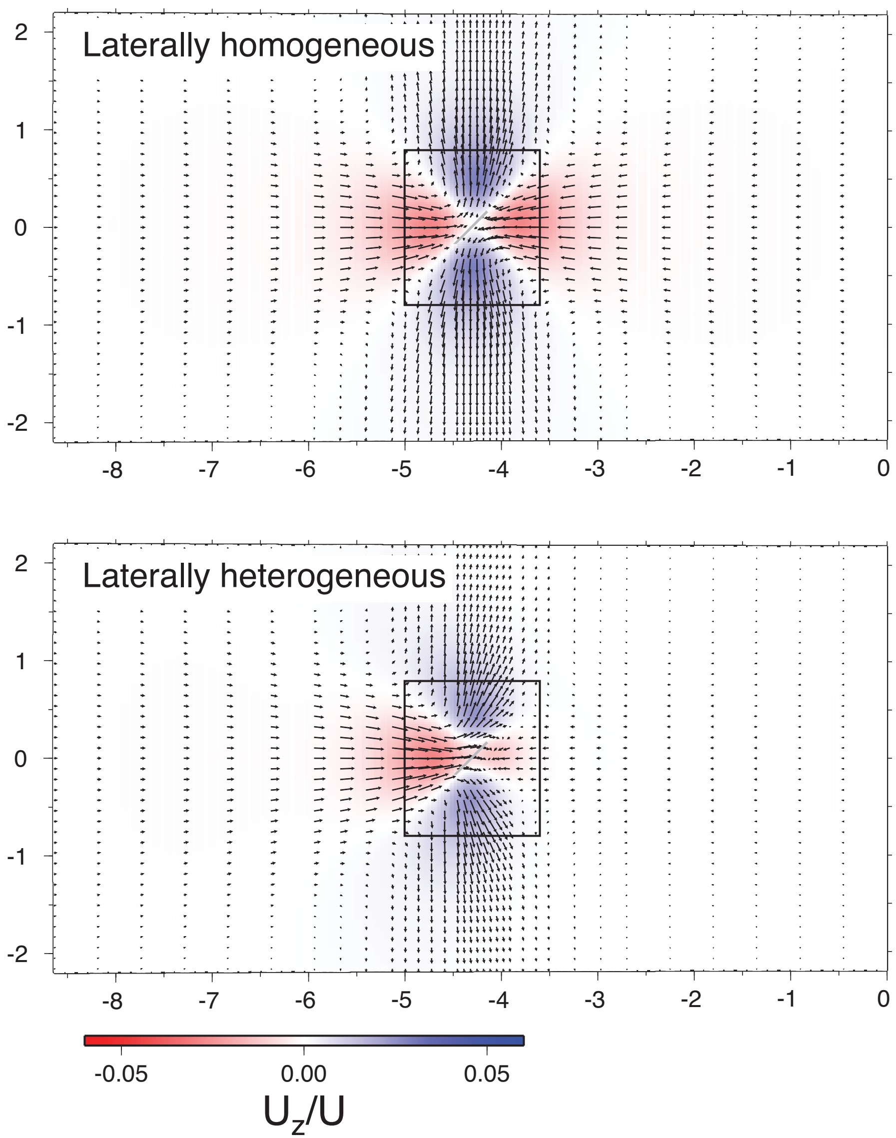
Cumulative displacements up to 5τ for post-dip-slip relaxation

Cumulative displacements along the equator for post-dip-slip relaxation

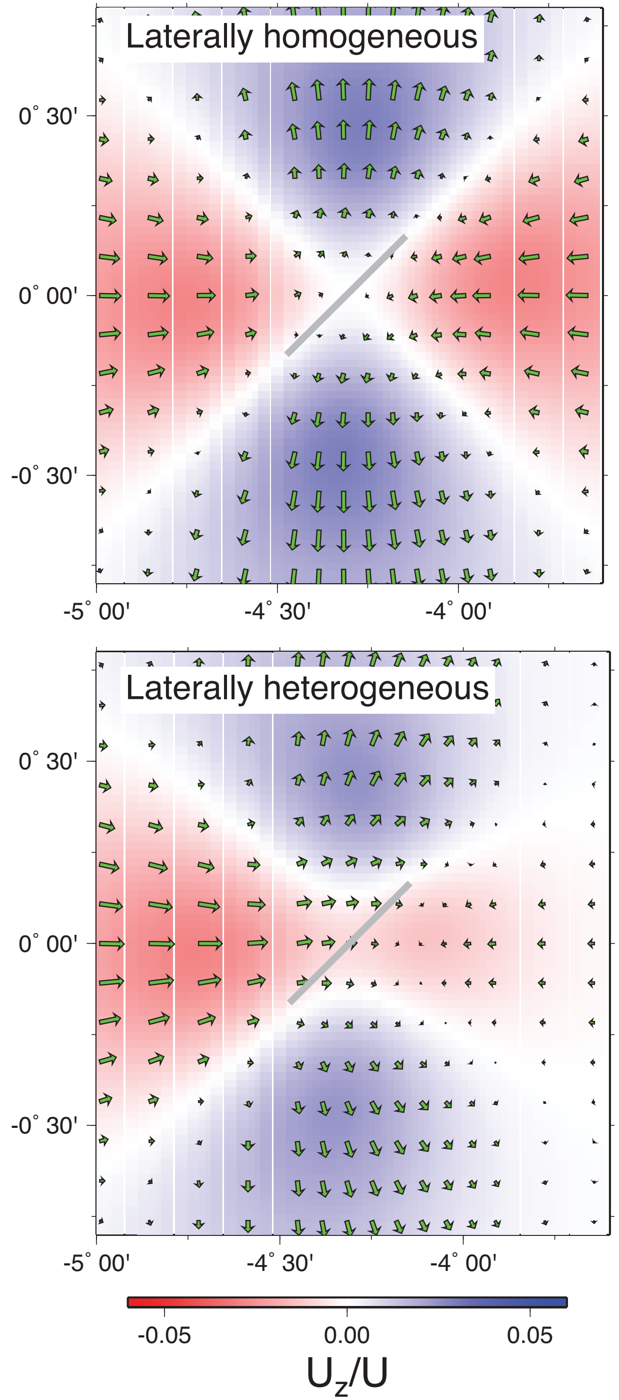


Comparison of Spectral Element Method with Analytic Solution



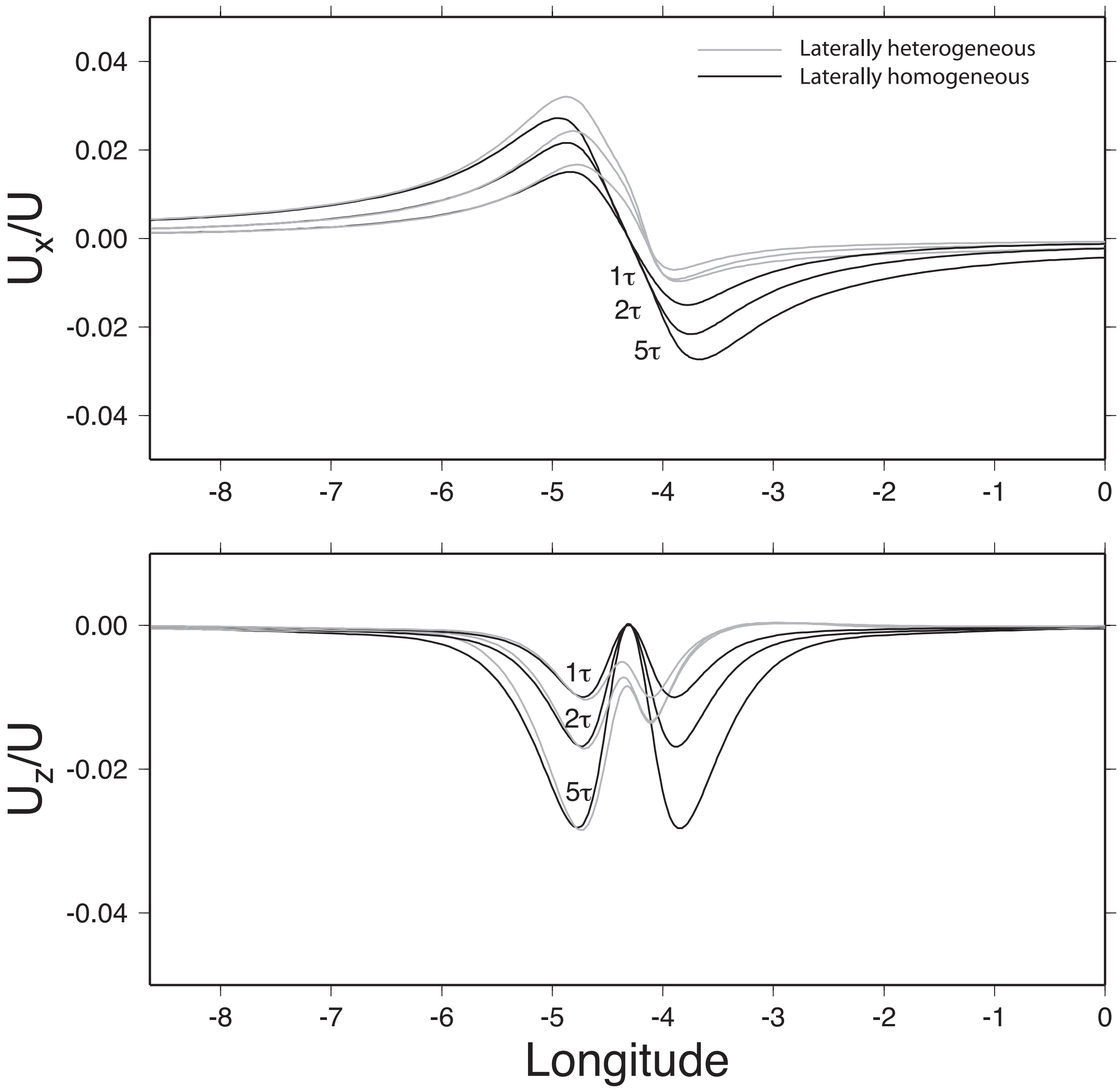


Cumulative displacements up to 5τ for post-strike-slip relaxation



Cumulative displacements up to 5τ for post-strike-slip relaxation

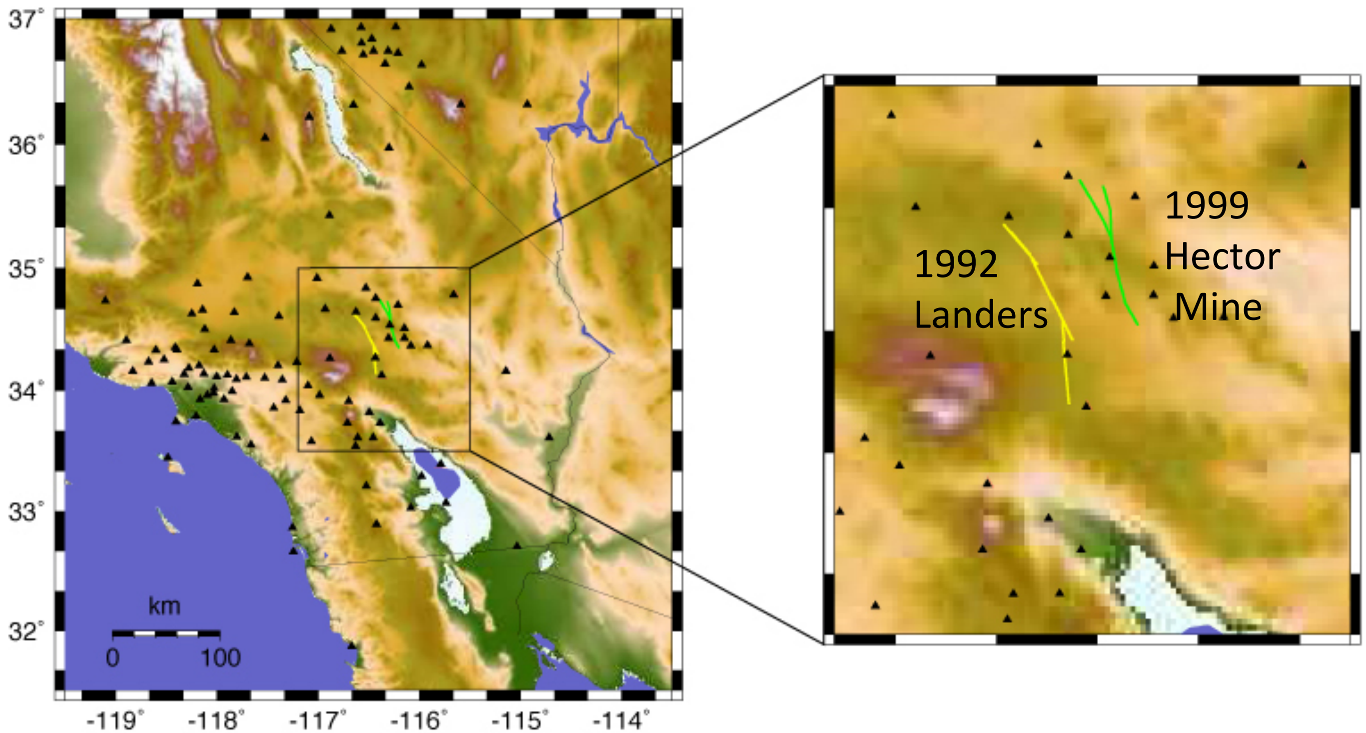
Cumulative displacements along the equator for post-strike-slip relaxation

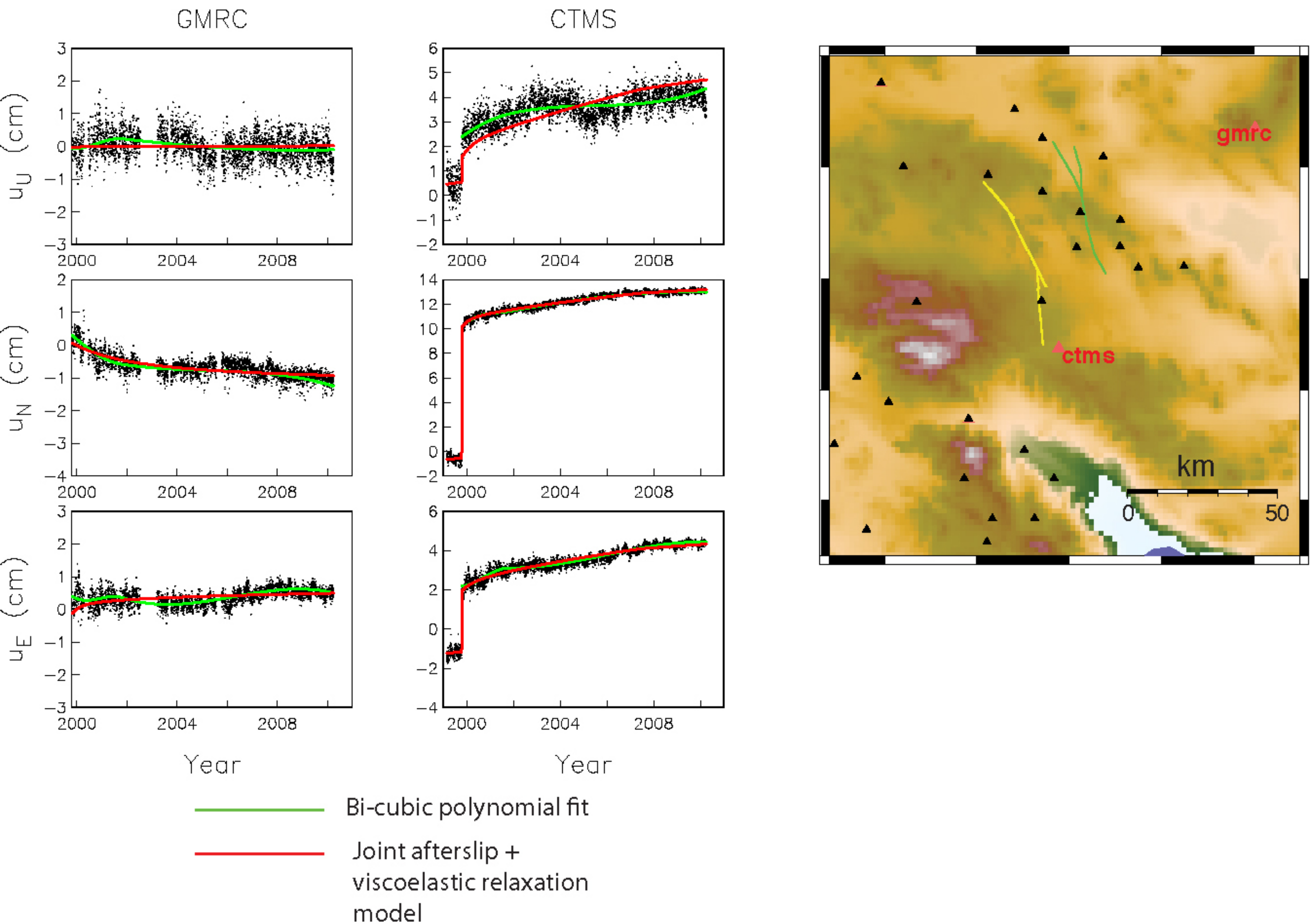


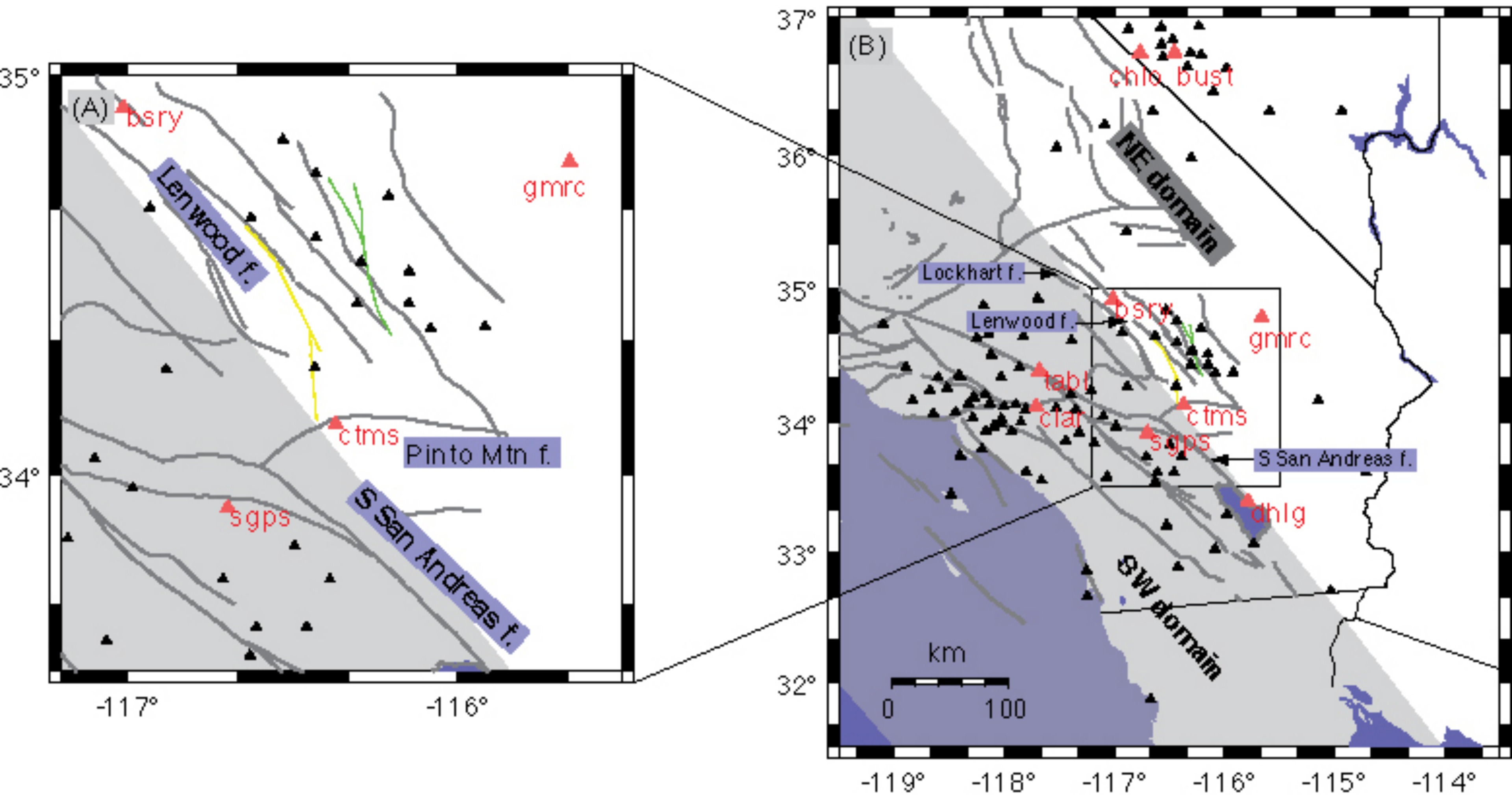
Application to post-Landers and Hector Mine relaxation

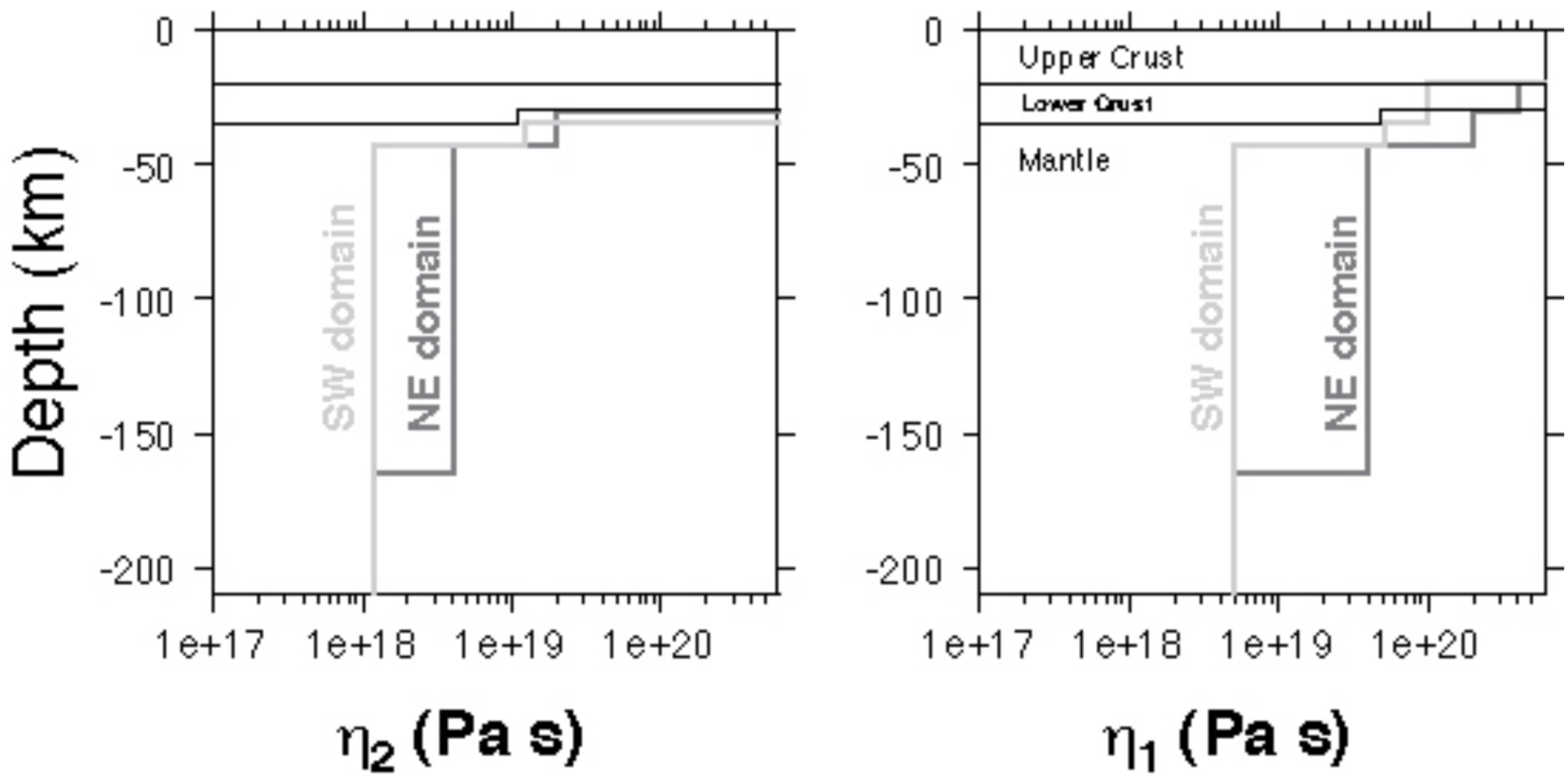
- Estimate transient velocities

- Joint afterslip + lower crust and mantle relaxation modeling
  -- 3D time-dependent displacements
  -- 2D viscoelastic structure

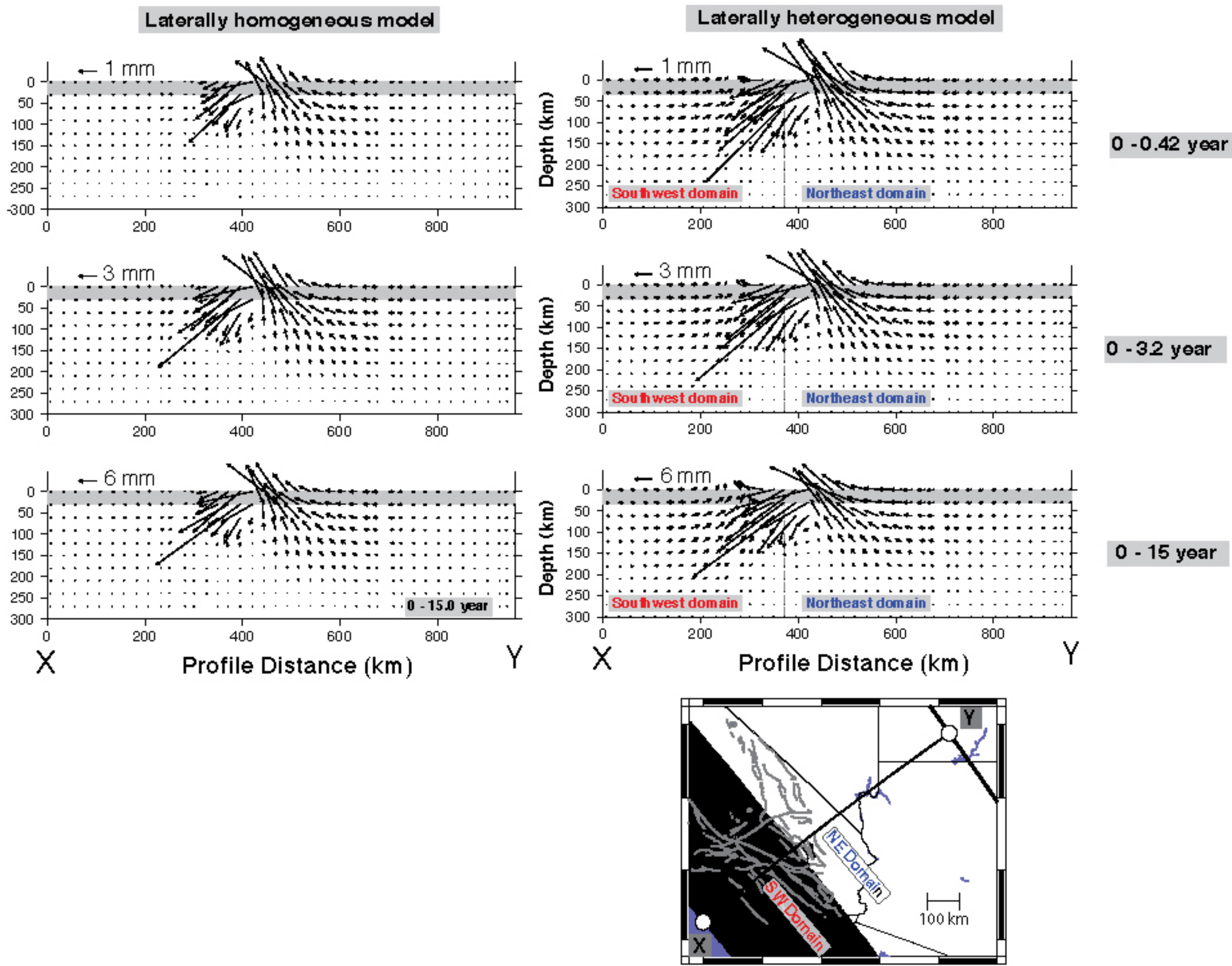



GMRC
CTMS
u_U (cm)
u_N (cm)
u_E (cm)
Year
Year
gmrc
ctms
km
0
50
Bi-cubic polynomial fit
Joint afterslip + viscoelastic relaxation model

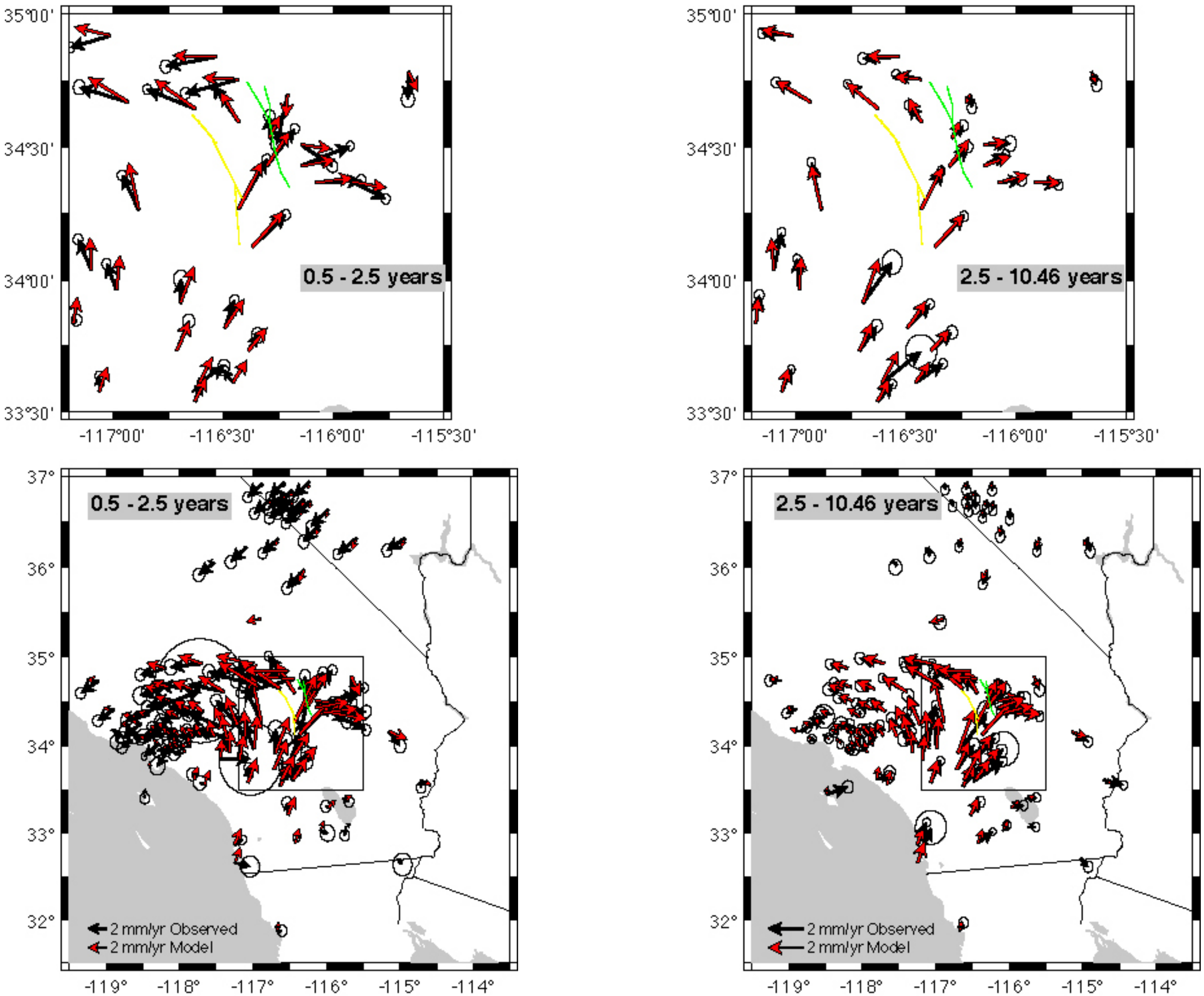

(A)
bsry
gmrc
Lenwood f.
ctms
Pinto Mtn f.
sgps
S San Andreas f.
35°
34°
-117°
-116°
(B)
chlo
bust
NE domain
Lockhart f.
Lenwood f.
bsry
gmrc
tabl
clar
ctms
sgps
S San Andreas f.
dhlg
SW domain
km
0
100
37°
36°
35°
34°
33°
32°
-119°
-118°
-117°
-116°
-115°
-114°



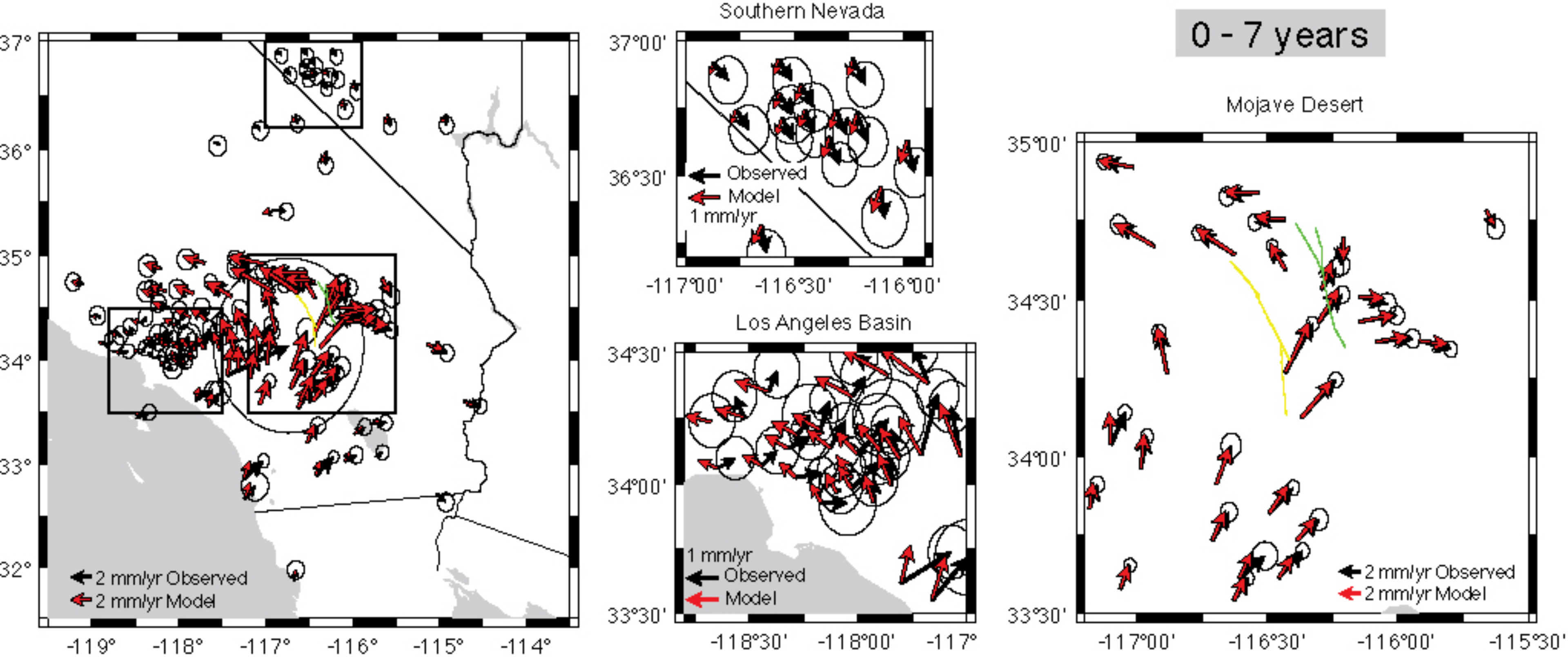
Depth (km)
0
-50
-100
-150
-200
SW domain
NE domain
1e+17
1e+18
1e+19
1e+20
η_2 (Pa s)
Upper Crust
Lower Crust
Mantle
η_1 (Pa s)



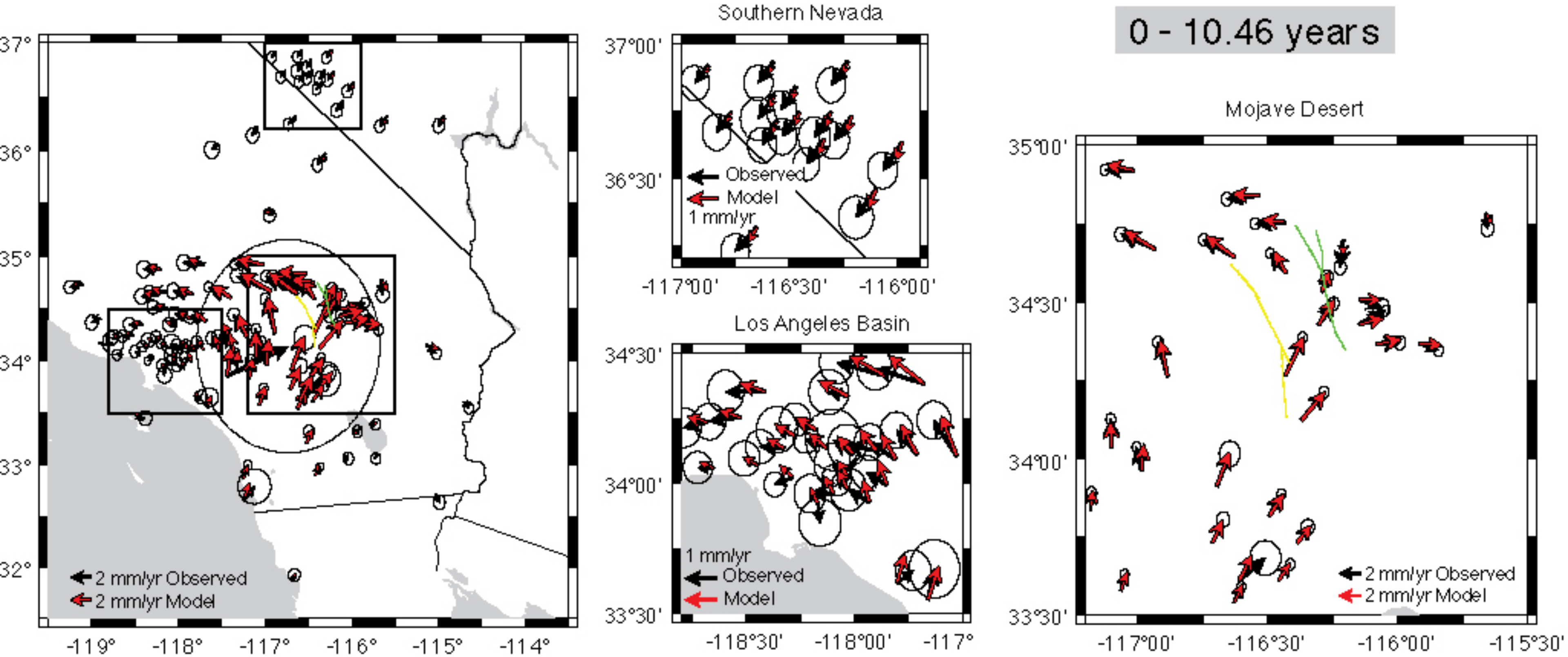
Laterally homogeneous model
Laterally heterogeneous model
1 mm
3 mm
6 mm
Depth (km)
Southwest domain
Northeast domain
0 - 0.42 year
0 - 3.2 year
0 - 15 year
0 - 15.0 year
X
Y
Profile Distance (km)
NE Domain
SW Domain
100 km



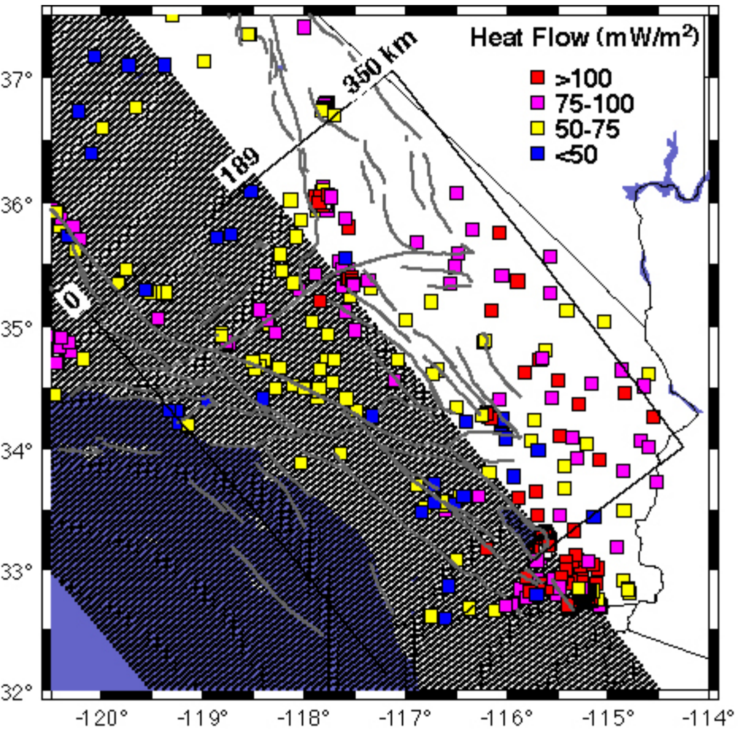
0.5 - 2.5 years
2.5 - 10.46 years
35°00'
34°30'
34°00'
33°30'
-117°00'
-116°30'
-116°00'
-115°30'
37°
36°
35°
34°
33°
32°
-119°
-118°
-117°
-116°
-115°
-114°
2 mm/yr Observed
2 mm/yr Model

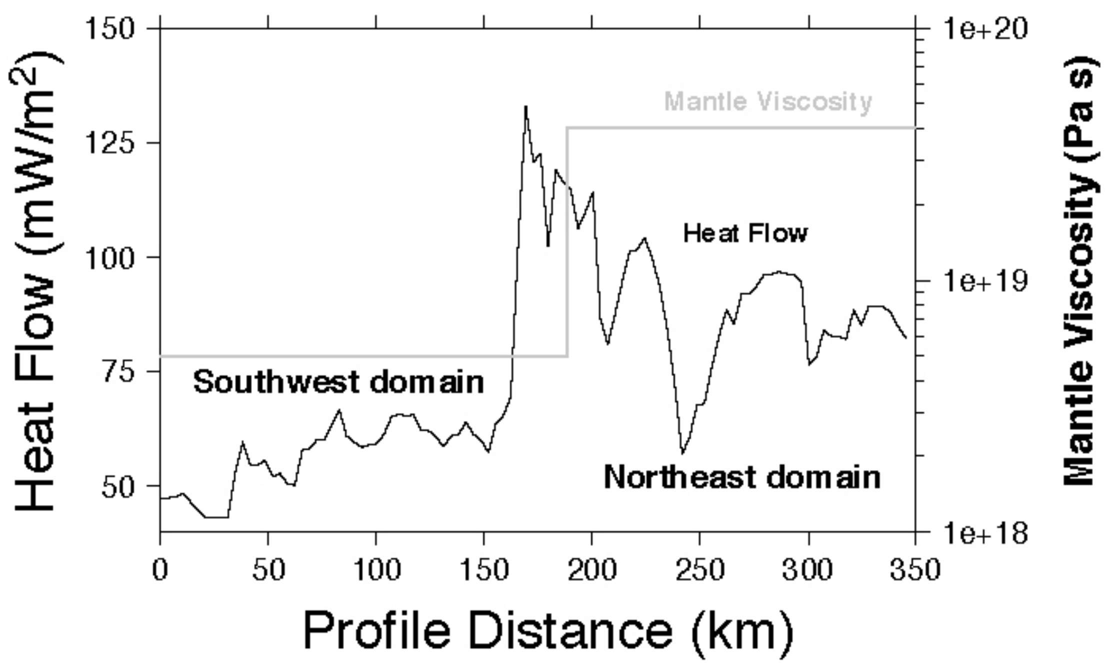


0 - 7 years
37°
36°
35°
34°
33°
32°
-119°
-118°
-117°
-116°
-115°
-114°
2 mm/yr Observed
2 mm/yr Model
Southern Nevada
37°00'
36°30'
-117°00'
-116°30'
-116°00'
Observed
Model
1 mm/yr
Los Angeles Basin
34°30'
34°00'
33°30'
-118°30'
-118°00'
-117°
1 mm/yr
Observed
Model
Mojave Desert
35°00'
34°30'
34°00'
33°30'
-117°00'
-116°30'
-116°00'
-115°30'
2 mm/yr Observed
2 mm/yr Model



0 - 10.46 years
Southern Nevada
Los Angeles Basin
Mojave Desert
2 mm/yr Observed
2 mm/yr Model
Observed
Model
1 mm/yr
37°
36°
35°
34°
33°
32°
-119°
-118°
-117°
-116°
-115°
-114°
37°00'
36°30'
-117°00'
-116°30'
-116°00'
34°30'
34°00'
33°30'
-118°30'
-118°00'
-117°
35°00'
-115°30'


Heat Flow (mW/m²)
>100
75-100
50-75
<50
350 km
189
0
37°
36°
35°
34°
33°
32°
-120°
-119°
-118°
-117°
-116°
-115°
-114°


Mantle Viscosity
Heat Flow
Southwest domain
Northeast domain
Heat Flow (mW/m²)
Mantle Viscosity (Pa s)
Profile Distance (km)
150
125
100
75
50
1e+20
1e+19
1e+18
0
50
100
150
200
250
300
350

Available code: VISCO2.5D

- 3D quasi-static displacement field on 2D viscoelastic structures
- Limited to linear rheologies
- Currently implements simple elements – bounded by spherical shells and vertical interfaces
- Simple to run in parallel