Multi-scale issues and two scale homogenization solutions for the direct and inverse problem in seismology.

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Introduction and motivations : forward modeling issues with SEM

SEM characteristics for the wave propagation direct problem :

- A finite element like method (flexible)
- convergence of pseudo-spectral methods (accurate)
- if based on hexahedric meshes + GLL quadrature : diagonal mass matrix $\Rightarrow$ explicit time scheme (efficient)

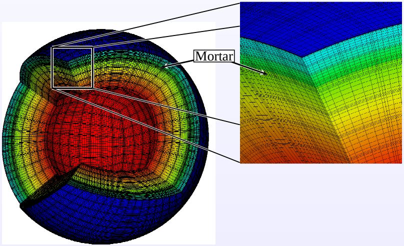
SEM constrains :

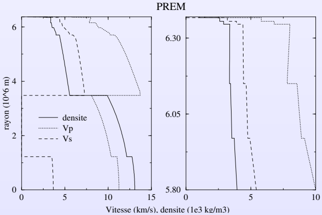
- Sampling : 2 wavelengths/element for a degree 8 ;
- mesh design : all physical discontinuities have to be meshed ;
- explicit time scheme stability : $dt \leq C \frac{dx_{min}}{V_{pmax}}$.

Introduction and motivations : forward modeling issues

Two simple examples of SEM meshes

Global scale





Local scale

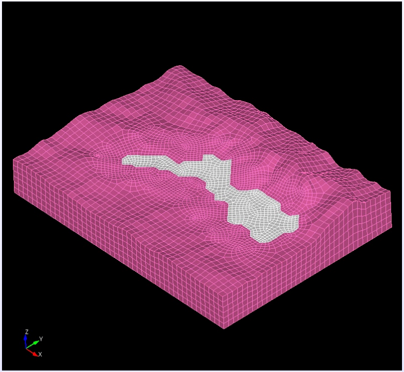


figure from E. Delavaud

Mesh design and constrain is a strong limiting factor for SEM based of hexahedric mesh. One can do better job with other technique (based of tetrahedric meshes ?), but will it solve everything ?

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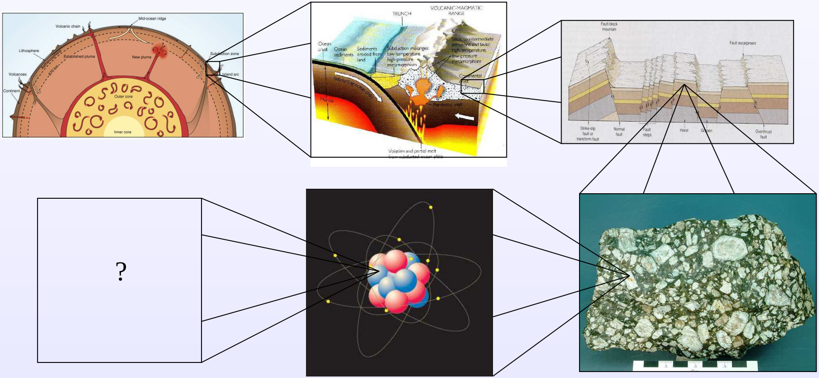
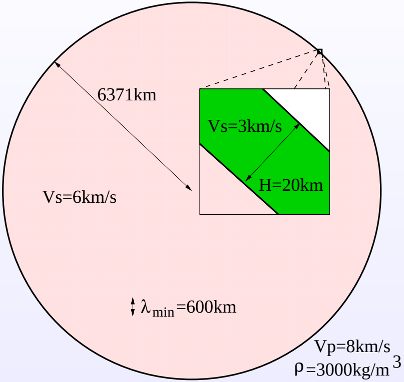
Down to what scale should we go ?



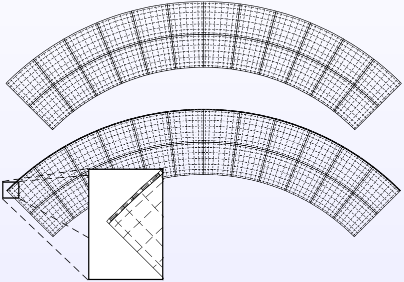
⇒ Scale separation and up-scaling is always an issue in realistic cases.

A limited aspect of this general problem : thin shallow layers

An homogeneous sphere with a thin slow layer on the top



Mesh 1 (no shallow layer)



Mesh 2 (with shallow layer)

Source corner period : 100 s

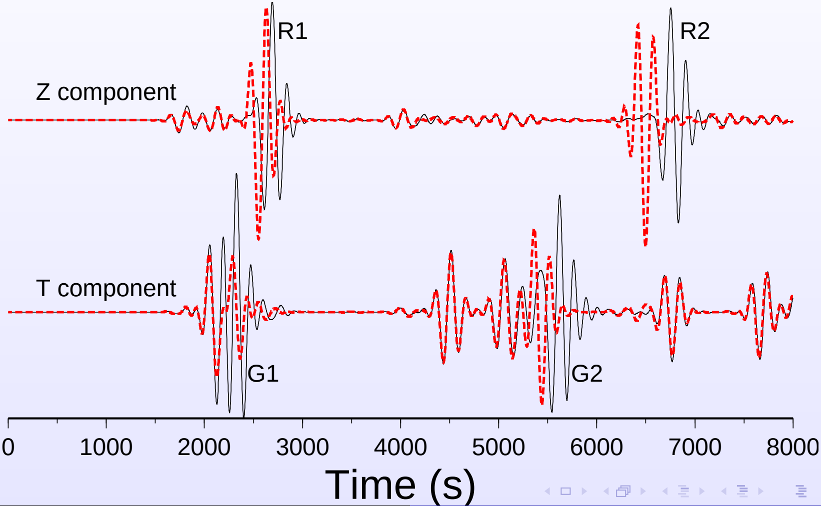
Just because of the stability condition, propagating in mesh 2 is 30 times more computing intensive than in mesh 1.

This is a generic case of crustal models (e.g. 3SMAC, CRUST2.0) implementation in SEM at the global scale.

Two classical solutions

Solution 1 : Ignoring the shallow layer

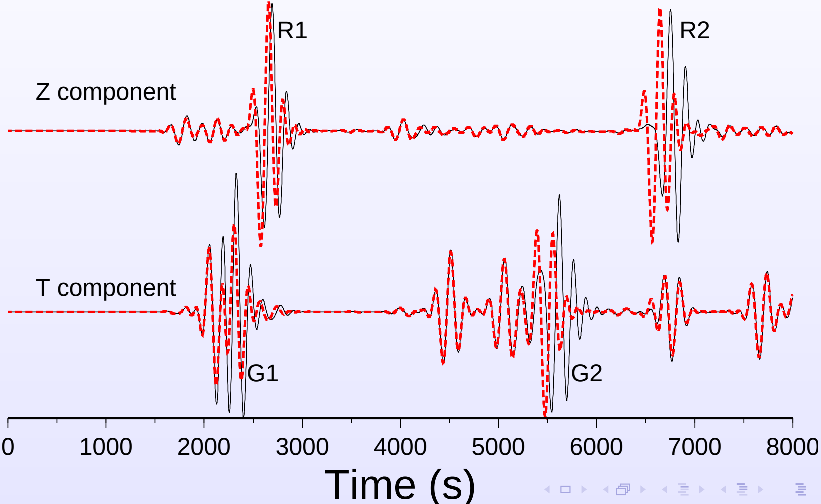
Black line : reference solution ; red line : approximate solution



Two classical solutions

Solution 2 : the discontinuity is not honored by the mesh

Black line : reference solution ; red line : approximate solution



A third solution : matching asymptotic expansions

Assumption : $\varepsilon = \frac{H}{\lambda_{min}} << 1$

New variable : $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$

expansion in the shallow layer :

$$\mathbf{u}_c^\varepsilon(\mathbf{y}) = \sum_i \varepsilon^i \mathbf{u}_c^i(\mathbf{y})$$

$$\boldsymbol{\sigma}_c^\varepsilon(\mathbf{y}) = \sum_i \varepsilon^i \boldsymbol{\sigma}_c^i(\mathbf{y})$$

$$\rho \ddot{\mathbf{u}}_c^\varepsilon - \nabla \cdot \boldsymbol{\sigma}_c^\varepsilon = \mathbf{f}$$

$$\boldsymbol{\sigma}_c^\varepsilon = \mathbf{c} : \epsilon(\mathbf{u}_c^\varepsilon)$$

$\mathbf{c}(\mathbf{y}) = \mathbf{c}^\varepsilon(\mathbf{x}/\varepsilon)$

Boundary condition : free surface

expansion in the volume :

$$\mathbf{u}^\varepsilon(\mathbf{x}) = \sum_i \varepsilon^i \mathbf{u}^i(\mathbf{x})$$

$$\boldsymbol{\sigma}^\varepsilon(\mathbf{x}) = \sum_i \varepsilon^i \boldsymbol{\sigma}^i(\mathbf{x})$$

$$\rho^s \ddot{\mathbf{u}}^\varepsilon - \nabla \cdot \boldsymbol{\sigma}^\varepsilon = \mathbf{f}$$

$$\boldsymbol{\sigma}^\varepsilon = \mathbf{c}^s : \epsilon(\mathbf{u}^\varepsilon)$$

$\mathbf{c}^s(\mathbf{x})$ is “smooth”

Boundary condition : regularity at the center of the earth

the solutions must match in region where both solutions are valid

- introducing the expansion in the waves equation
- using $\frac{\partial}{\partial \mathbf{x}} \longrightarrow \frac{1}{\varepsilon}\frac{\partial}{\partial \mathbf{y}}$
- identifying terms in ε^i

a series of equations is obtained that can be solved one by one

- order 0 : regular wave equation with free boundary condition in the "smooth" model ($\mathbf{c}^s$)
- order i ($i > 0$) : regular wave equation in the "smooth" model ($\mathbf{c}^s$) but with a special DtN boundary condition.

At the order 2, on the surface :

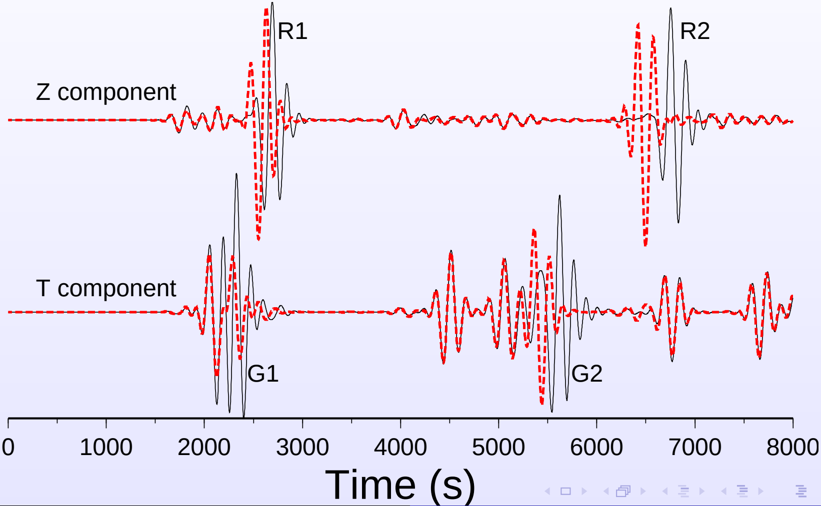
$$\mathbf{t} = \varepsilon \left\{ \; X^1_\rho \ddot{\mathbf{u}} - X^1_{a1} \; \boldsymbol{\nabla}_1 (\boldsymbol{\nabla}_1 \cdot \mathbf{u}_1) + X^1_N \; \boldsymbol{\nabla}_1 \times \boldsymbol{\nabla}_1 \times \mathbf{u}_1 \; \right\}$$
$$+\varepsilon^2 \left\{ \; X^2_{a1} \left(\nabla^2_1 (\boldsymbol{\nabla}_1 \cdot \mathbf{u}_1) \, \hat{\mathbf{z}} - \boldsymbol{\nabla}_1 \mathrm{u}_z \; \right) + X^2_b \; \left((\boldsymbol{\nabla}_1 \cdot \ddot{\mathbf{u}}_1) \, \hat{\mathbf{z}} - \boldsymbol{\nabla}_1 \ddot{\mathrm{u}}_z \right) \; \right\}$$

$$\text{with (e.g.)} \quad X^1_\rho = -\int_0^{\frac{H}{\varepsilon}} (\rho(y) - \rho^s(a)) \, dy$$

Order 0 matching asymptotic result

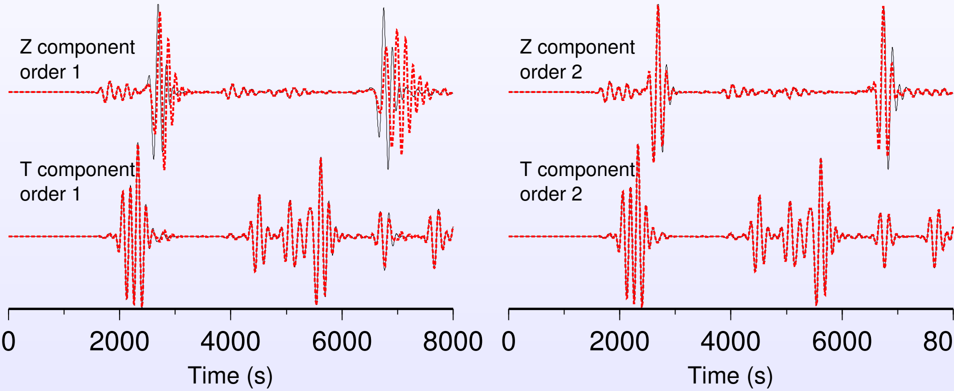
Same as "ignoring the shallow layer" solution

Black line : reference solution ; red line : approximate solution

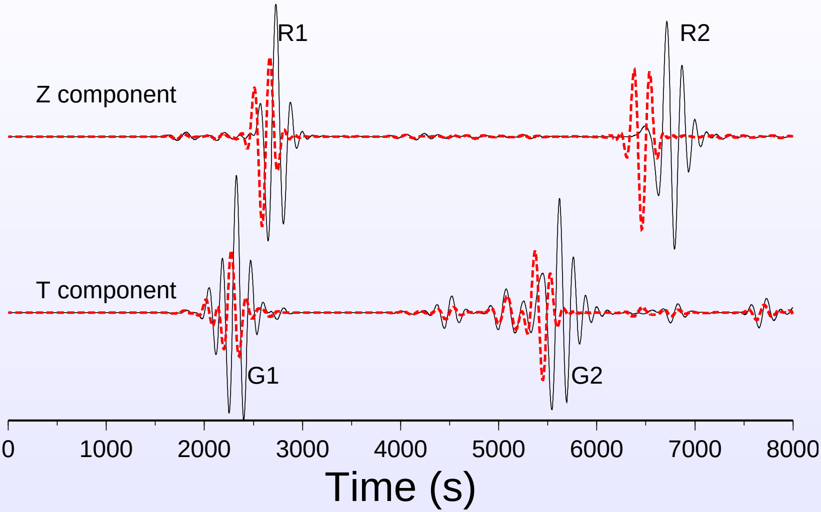


Order two matching asymptotic results

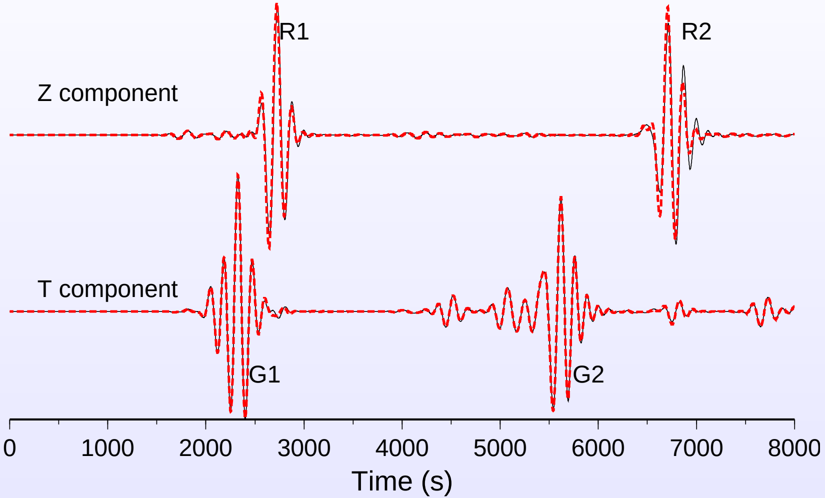
Black line : reference solution ; red line : order 1 or 2 solution



If the source is located in the shallow layer



If the source is located in the shallow layer : order 2 solution with order 1 correction for the source



Conclusions on matching asymptotic expansions for shallow layers

Interest

It gives a macroscopic view of small scale just bellow the surface. It is useful for

- forward modeling technique (e.g accurate crust model implementation) ;
- seismic imaging technique (e.g. can solve the global scale crustal correction issues).

Limitations

- the frequency band of accuracy is fixed by the thickness of the shallow layer ;
- the DtN can lead to instabilities (but this can be worked out) ;
- it doesn't solve the problem of deep small scales.

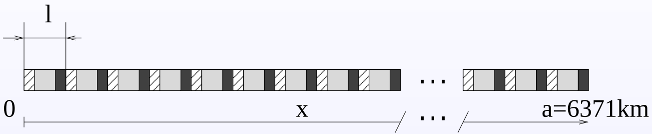
To move to a more general case, we need two scale homogenization

Two scale homogenization for layered media

- Two scale homogenization is widely used in mechanics to assess effective macroscopic behavior of microscopic periodic structures.
- it can also be (and has been) applied to the dynamic case

For this presentation, we limit our work to layered media (or smooth lateral variation) ; we start with periodic media, but it can be extended to non-periodic media.

Homogenization principle for the wave equation for a simple case : P wave in a bar



Wave equations and boundary conditions

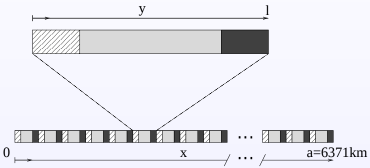
$$\rho \ddot{u} + \sigma_{,x} = f$$
$$\sigma = E u_{,x}$$

$$\sigma(0) = 0; \sigma(a) = 0;$$

Main assumption

$$\varepsilon = \frac{\ell}{\lambda_{min}} << 1$$

Homogenization principle in a 1D bar



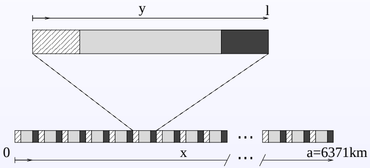
A new variable y

$$y = \frac{x}{\epsilon}$$

When $\varepsilon \longrightarrow 0$, x and y are assumed independent.

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y}$$

Homogenization principle in a 1D bar



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$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y}$$

All quantities (ρ, E, u, σ ...) are assumed periodic in y
Cell average properties : for all $g(x,y)$ we have :

- $\langle g_{,y} \rangle = 0$
- $g_{,y} = 0 \Rightarrow g(x,y) = g(x) = \langle g \rangle$
- $g = \langle g \rangle + \bar{g}$.

$$\langle g \rangle = \frac{\varepsilon}{\ell}\int_0^{\frac{\ell}{\varepsilon}} g(x,y)dy$$

Homogenization principle in a 1D bar

We introduce :

$$u^\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u^i(x, y = \frac{x}{\varepsilon})$$

$$\sigma^\epsilon(x) = \sum_{i=-1}^{+\infty} \epsilon^i \sigma^i(x, y = \frac{x}{\varepsilon})$$

The wave equations become, $\forall i$:

$$\rho \ddot{u}^i + \sigma^i_{,x} + \sigma^{i+1}_{,y} = f\delta_{i0}$$

$$\sigma^i = E(u^i_{,x} + u^{i+1}_{,y})$$

$$\sigma^i(a) = 0$$

Using the cell average properties

- $i = -2$ gives $\sigma^{-1} = \langle \sigma^{-1} \rangle$
- $i = -1$ gives $\sigma^{-1} = 0$; $\sigma^0 = \langle \sigma^0 \rangle$; $u^0 = \langle u^0 \rangle$

Homogenization principle in a 1D bar

- $i = 0$ gives the order 0 homogenized wave equation :

$$\langle\rho\rangle\ddot{u}^0 + \sigma^0_{,x} = f$$
$$\langle\frac{1}{E}\rangle\sigma^0 = u^0_{,x}$$
$$\sigma^0(a) = 0$$

and the order 1 periodic corrector :

$$u^1_{,y} = \chi^a_{,y}\sigma^0 \quad \text{with} \quad \chi^a_{,y} = 1/E - \langle 1/E\rangle$$

$$\sigma^1_{,y} = \chi^b_{,y}\ddot{u}^0 \quad \text{with} \quad \chi^b_{,y} = \langle\rho\rangle - \rho$$

with the condition $\langle\chi^x\rangle = 0$, then

$$\bar{u}^1(x,y) = \chi^a(y)\sigma^0(x)$$
$$\bar{\sigma}^1(x,y) = \chi^b(y)\ddot{u}^0(x)$$

$\langle u^1\rangle$ and $\langle\sigma^1\rangle$ remain to be found to obtain a complete order 1 homogenized solution.

Homogenization principle in a 1D bar

- $i = 1$ gives

$$\langle\rho\rangle\langle\ddot{u}^1\rangle + \langle\sigma^1\rangle_{,x} = f - \langle\rho\chi^a\rangle\sigma^0$$

$$\langle\frac{1}{E}\rangle\langle\sigma^1\rangle = \langle u^1\rangle_{,x} - \langle\frac{\chi^b}{E}\rangle\ddot{u}^0$$

$$\langle\sigma^1\rangle(a) = -\bar{\sigma}^1(a) = \chi^b\,(a/\varepsilon)\,\ddot{u}^0(a)$$

Homogenization principle in a 1D bar

- $i = 1$ gives

$$\langle\rho\rangle\langle\ddot{u}^1\rangle + \langle\sigma^1\rangle_{,x} = f - \langle\rho\chi^a\rangle\sigma^0$$
$$\langle\frac{1}{E}\rangle\langle\sigma^1\rangle = \langle u^1\rangle_{,x} - \langle\frac{\chi^b}{E}\rangle\ddot{u}^0$$
$$\langle\sigma^1\rangle(a) = -\bar{\sigma}^1(a) = \chi^b\left(a/\varepsilon\right)\ddot{u}^0(a)$$

Now, using $u = u^0 + \epsilon u^1$ and $\sigma = \sigma^0 + \epsilon\sigma^1$:

First order homogenized wave equations

$$\langle\rho\rangle\langle\ddot{u}\rangle + (1 + \epsilon\langle\rho\chi^a\rangle)\langle\sigma\rangle_{,x} = f$$
$$\langle\frac{1}{E}\rangle\langle\sigma\rangle = \langle u\rangle_{,x} - \epsilon\langle\frac{\chi^b}{E}\rangle\langle\ddot{u}\rangle$$
$$\langle\sigma\rangle(a) = -\varepsilon\chi^b(a/\varepsilon)\langle\ddot{u}\rangle(a)$$
$$u(x) = \langle u\rangle(x) + \varepsilon\langle\sigma\rangle(x)\chi^a(\frac{x}{\varepsilon}))$$

Homogenization principle in a 1D : summary

Original equations

$$\rho \ddot{u} + \sigma_{,x} = f$$
$$\sigma = E u_{,x}$$

$$\sigma(a) = 0$$

0 order homogenized wave equations

$$\langle \rho \rangle \langle \ddot{u} \rangle + \langle \sigma \rangle_{,x} = f$$
$$\langle \sigma \rangle = \frac{1}{\langle 1/E \rangle} \langle u \rangle_{,x}$$

$$\langle \sigma \rangle (a) = 0$$

$$u(x) = \langle u \rangle (x) ; \sigma(x) = \langle \sigma \rangle (x)$$

Remarque : $u_{,x} = \frac{\sigma(x)}{E} = \frac{\langle \sigma \rangle (x)}{E} \neq \langle u \rangle_{,x}$!

Homogenization principle in a 1D : summary

Original equations

$$\rho \ddot{u} + \sigma_{,x} = f$$
$$\sigma = E u_{,x}$$

$$\sigma(a) = 0$$

First order homogenized wave equations

$$\chi^a_{,y} = 1/E - \langle 1/E \rangle; \quad \chi^b_{,y} = \langle \rho \rangle - \rho$$

$$\langle \rho \rangle \langle \ddot{u} \rangle + (1 + \epsilon \langle \rho \chi^a \rangle) \langle \sigma \rangle_{,x} = f$$
$$\langle \sigma \rangle = \frac{1}{\langle 1/E \rangle} \langle u \rangle_{,x} - \epsilon \frac{1}{\langle 1/E \rangle} \langle \frac{\chi^b}{E} \rangle \langle \ddot{u} \rangle$$

$$\langle \sigma \rangle (a) = -\varepsilon \chi^b (a/\varepsilon) \langle \ddot{u} \rangle (a)$$

$$u(x) = \langle u \rangle (x) + \varepsilon \langle \sigma \rangle (x) \chi^a (\frac{x}{\varepsilon})$$

Homogenization principle for layered media

- for $i = 0$: the order 0 homogenized equation is obtained,

$$\rho \ddot{\mathbf{u}}^0 - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^0 = \mathbf{f}$$
$$\boldsymbol{\sigma}^0 = \tilde{\mathbf{c}} : \epsilon(\mathbf{u}^0)$$

  and free boundary condition. $\tilde{\mathbf{c}}$ (for the transverse isotropic case) is such

$$\tilde{N} = \langle N \rangle, \; \frac{1}{\tilde{L}} = \langle \frac{1}{L} \rangle, \; \frac{1}{\tilde{C}} = \langle \frac{1}{C} \rangle$$
$$\frac{\tilde{F}}{\tilde{C}} = \langle \frac{F}{C} \rangle, \; \tilde{A} - \frac{\tilde{F}^2}{\tilde{C}} = \langle A - \frac{F^2}{C} \rangle$$

  This is the Backus (1962)'s result. $i = 0$ also provide the order 0 source correction and the first order correctors.

- order > 0 gives homogenized wave equation with a different Hook law and different boundary conditions (DtN).

Order 0 correction for the source :

$$M_{rr}^{h0} = M_{rr} \frac{C^{h0}}{C}$$

$$M_{\theta\theta}^{h0} = M_{\theta\theta} + M_{rr} \frac{F^{h0} - F}{C}$$

$$M_{\phi\phi}^{h0} = M_{\phi\phi} + M_{rr} \frac{F^{h0} - F}{F}$$

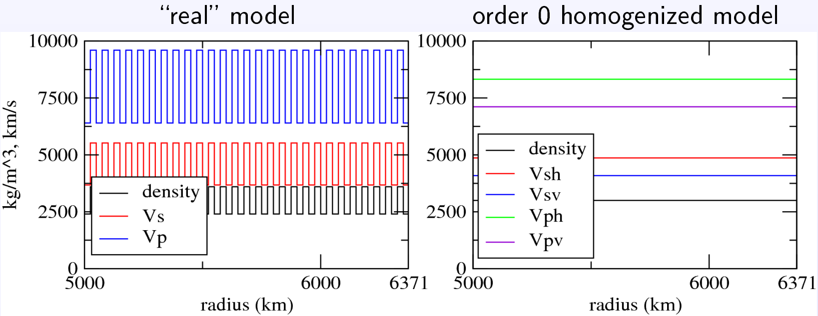
$$M_{r\theta}^{h0} = M_{r\theta} \frac{L^{h0}}{L}$$

$$M_{r\phi}^{h0} = M_{r\phi} \frac{L^{h0}}{L}$$

$$M_{\theta\phi}^{h0} = M_{\theta\phi}$$

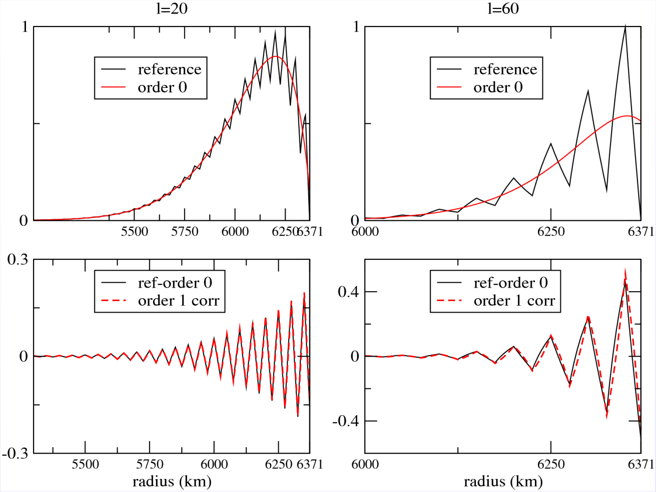
Homogenization in layered earth : examples

Example in a periodic layered model :



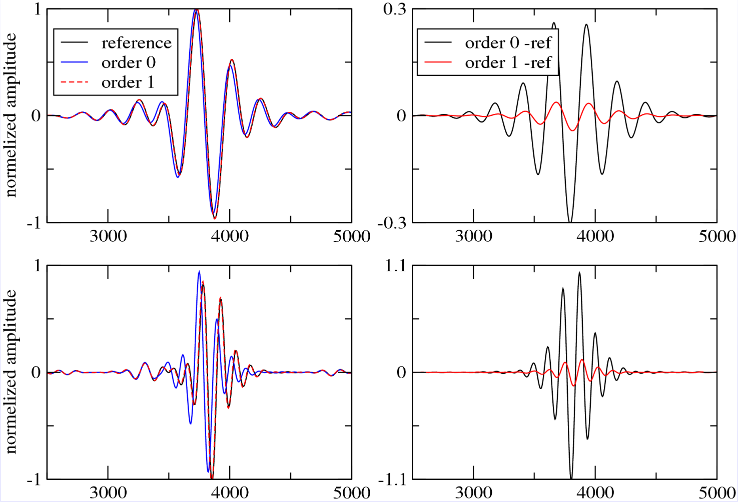
Homogenization in layered earth : examples

0S20 and 0S60 modes m5 minor (traction)



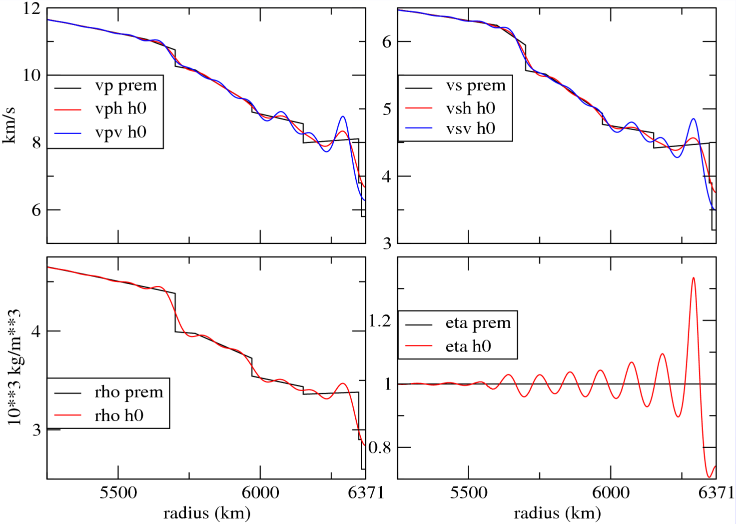
Homogenization in layered earth : examples

traces :



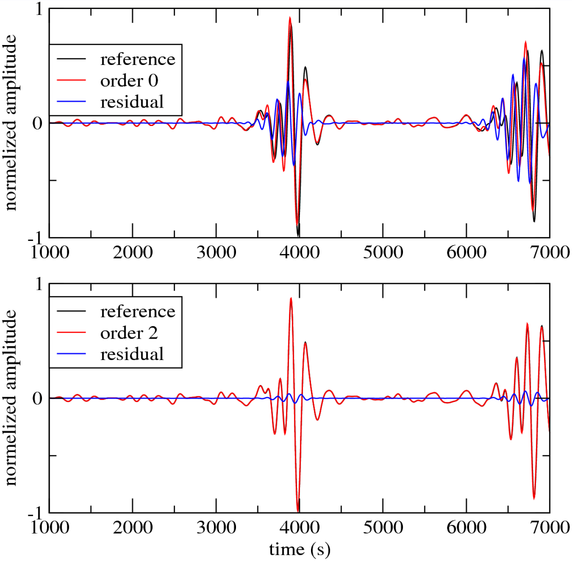
Homogenization in layered earth : examples

Example in PREM (non periodic model !)



Homogenization in layered earth : examples

A seismogram in PREM :



Does it really matter ?

For the direct problem :

- Yes, if the model contains small scales ;
- No, if the model is simple enougth.

For the imaging problem :

- Yes, almost always : whatever the frequency band used of the data, it is likely that scales much smaller than the minimum wavelength exist in the real model.

Conclusions

For the layered media case, matching asymptotic expansions and homogenization allow up-scaling laws. This implies non trivial things like modifying the boundary condition and the wave equation. On the other hand this allow forward modeling in a simpler medium. For the inverse problem, it allows to build a parametrization consistent with the frequency band of the data.

Perspectives

1. to apply this work to laterally smoothly varying media for :
   - forward modeling issue (e.g. using crustal model like 3SMAC or CRUST2.0)
   - tomographic inverse problem
   - moment tensor inversion
2. Extension to full 3D problem (rapid variation in all directions) : Laurent Guillot.
3. Finally, application to full waveform inversion with the Spectral Elements.