Accuracy of Spectral Element Methods for Wave Propagation Modeling

Saulo P. Oliveira and Géza Seriani
Istituto Nazionale di Oceanografia e di Geofisica Sperimentale, Trieste, Italy

Summary – We review a recent methodology for estimating the numerical dispersion of spectral element methods with arbitrary order for 1D to 3D seismic wave propagation problems. This approach circumvents the issue of spurious modes of propagation and reduces to simple 1D calculations in Cartesian grids. We use this approach to select the combination of consistent and lumped mass matrices that yields the lowest dispersion error and to study numerical dispersion caused by mesh distortion.

Standard Analysis

Assume an infinite, periodical mesh and homogeneous media

$$\ddot{u} - c^2\Delta u = 0 \;\Rightarrow\; u \leftarrow e^{-i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{x})} \;\Rightarrow\; \omega = c|\boldsymbol{\kappa}|$$

Plug plane wave into discrete equations; the amplitude depends on the mesh nodes that define the mesh periodicity

$$\boldsymbol{M}\ddot{\boldsymbol{u}}^* + c^2\boldsymbol{K}\boldsymbol{u}^* = \boldsymbol{0} \;\Rightarrow\; u_p^* \leftarrow A_p e^{-i(\omega^* t - \boldsymbol{\kappa}\cdot\boldsymbol{x}_p)} \;\Rightarrow\; \overline{\boldsymbol{K}}\boldsymbol{A}^* = \chi\overline{\boldsymbol{M}}\boldsymbol{A}, \quad \chi = (\omega^*/c)^2$$

Eigenvalue problem yields multiple dispersion relations

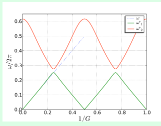


Fig 1: dispersion of the 1D quadratic element [1]

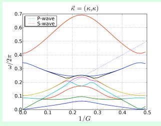


Fig 2: dispersion of the elastic 2D cubic element [4]

Rayleigh Quotient Approximation

Let us now consider a plane wave with constant amplitude:

$$\boldsymbol{M}\ddot{\boldsymbol{u}}^* + c^2\boldsymbol{K}\boldsymbol{u}^* = \boldsymbol{0} \;\Rightarrow\; u_p^* \leftarrow e^{-i(\omega^* t - \boldsymbol{\kappa}\cdot\boldsymbol{x}_p)} \;\Rightarrow\; \boldsymbol{K}\boldsymbol{w} = \chi\boldsymbol{M}\boldsymbol{w}, \quad w_p^* \leftarrow e^{i\boldsymbol{\kappa}\cdot\boldsymbol{x}_p}$$

We can't always find ω^* because $\boldsymbol{w}$ may not be an eigenvector, but its Rayleigh quotient approximation is well defined and unique

$$\chi = \frac{\overline{\boldsymbol{w}}^T\boldsymbol{K}\boldsymbol{w}}{\overline{\boldsymbol{w}}^T\boldsymbol{M}\boldsymbol{w}} \;\Rightarrow\; \omega^* = c\sqrt{\frac{\overline{\boldsymbol{w}}^T\boldsymbol{K}\boldsymbol{w}}{\overline{\boldsymbol{w}}^T\boldsymbol{M}\boldsymbol{w}}}$$

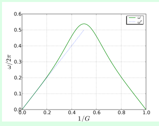


Fig 3: RQ dispersion of the 1D quadratic element

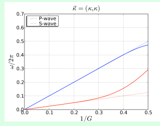


Fig 4: RQ dispersion of the elastic 2D cubic element

Case 1: 1D Acoustic

Assume a uniform mesh with coordinates

$$x_p = (e + \zeta_j)h, \quad h = \frac{1}{n_e}, \quad p = j + eN$$

$e = 1\ldots n_e$: elements
$\zeta_j, j = 0\ldots N$: collocation points in $[0,1]$
Chebyshev: $\zeta_j = [1 - \cos(\pi j/N)]/2$
Legendre : $P_N'(\zeta_j) = 0$ [3]

Restrict $\boldsymbol{w}$ to an element:

$$\boldsymbol{w}^e = \exp(i\kappa e h)\boldsymbol{v}, \quad \boldsymbol{v} = (\exp(i\kappa\zeta_0 h), \exp(i\kappa\zeta_1 h), \ldots, \exp(i\kappa\zeta_N h))$$

Element Matrices

$$\boldsymbol{M}^e = \frac{h}{2}\boldsymbol{A}, \quad A_{i,j} = \int_{-1}^{1}\phi_j(z)\phi_i(z)\,dz \qquad \boldsymbol{K}^e = \frac{2}{h}\boldsymbol{B}, \quad B_{i,j} = \int_{-1}^{1}\frac{\partial\phi_j}{\partial z}(z)\frac{\partial\phi_i}{\partial z}(z)\,dz$$

Element-by-element computation

$$\overline{\boldsymbol{w}}^T\boldsymbol{K}\boldsymbol{w} = \sum_{e=0}^{n_e-1}\overline{\boldsymbol{w}^e}^T\boldsymbol{K}^e\boldsymbol{w}^e = \frac{2}{h}\sum_{e=0}^{n_e-1}\exp(-i\kappa eh)\overline{\boldsymbol{v}}^T\boldsymbol{B}\exp(i\kappa eh)\boldsymbol{v} = \frac{2}{h}n_e\overline{\boldsymbol{v}}^T\boldsymbol{B}\boldsymbol{v}$$

$$\overline{\boldsymbol{w}}^T\boldsymbol{M}\boldsymbol{w} = \frac{h}{2}n_e\overline{\boldsymbol{v}}^T\boldsymbol{A}\boldsymbol{v}$$

The estimate reduces to

$$\omega^* = \frac{2c}{h}\sqrt{\frac{\overline{\boldsymbol{v}}^T\boldsymbol{B}\boldsymbol{v}}{\overline{\boldsymbol{v}}^T\boldsymbol{A}\boldsymbol{v}}}$$

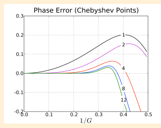


Fig 5: phase error: 1D elements of degree N=1,...,12 [5]

Case 2: 2D Elastic, Isotropic

Assume a square mesh with coordinates

$$(x_p, y_p) = (\,(e_1 + \zeta_{j_1})h, (e_2 + \zeta_{j_2})h\,), \quad p = p_1 + p_2 N n_e, \quad p_\alpha = j_\alpha + e_\alpha N, \quad e = e_1 + e_2 n_e$$

Discrete system of equations:

$$\begin{cases}\rho\boldsymbol{M}\ddot{\boldsymbol{u}}_1^* + \boldsymbol{K}_1\boldsymbol{u}_1^* + \boldsymbol{K}_2\boldsymbol{u}_2^* = \boldsymbol{0}\\ \rho\boldsymbol{M}\ddot{\boldsymbol{u}}_2^* + \boldsymbol{K}_2^T\boldsymbol{u}_1^* + \boldsymbol{K}_3\boldsymbol{u}_2^* = \boldsymbol{0}\end{cases}$$

$$\begin{aligned}\boldsymbol{M}^e &= (h^2/4)\boldsymbol{A}\otimes\boldsymbol{A}\\ \boldsymbol{K}_1^e &= E\boldsymbol{A}\otimes\boldsymbol{B} + \mu\boldsymbol{B}\otimes\boldsymbol{A}\\ \boldsymbol{K}_2^e &= \lambda\boldsymbol{C}^T\otimes\boldsymbol{C} + \mu\boldsymbol{C}\otimes\boldsymbol{C}^T\\ \boldsymbol{K}_3^e &= \mu\boldsymbol{A}\otimes\boldsymbol{B} + E\boldsymbol{B}\otimes\boldsymbol{A}.\end{aligned}$$

$$C_{i,j} = \int_{-1}^{1}\phi_j(z)\frac{\partial\phi_i}{\partial z}(z)\,dz$$

Discrete plane-wave solution:

$$\boldsymbol{u}_\alpha^* \leftarrow R_\alpha^* e^{-i\omega^* t}\boldsymbol{w}$$

Rayleigh quotient appoximation:

$$\begin{bmatrix} d_1 & d_2\\ \bar{d}_2 & d_3\end{bmatrix}\begin{bmatrix}R_1^*\\ R_2^*\end{bmatrix} = \chi\begin{bmatrix}R_1^*\\ R_2^*\end{bmatrix}, \quad d_i = \frac{\overline{\boldsymbol{w}}^T\boldsymbol{K}_i\boldsymbol{w}}{\overline{\boldsymbol{w}}^T\boldsymbol{M}\boldsymbol{w}}.$$

Restrict $\boldsymbol{w}$ to an element:

$$\boldsymbol{w}^e = e^{i(\kappa_1 e_1 + \kappa_2 e_2)}\boldsymbol{v}_2\otimes\boldsymbol{v}_1, \quad \boldsymbol{v}_\alpha = (\exp(i\kappa_\alpha\zeta_0 h), \ldots, \exp(i\kappa_\alpha\zeta_N h))$$

$$d_1 = \frac{4}{\rho h^2}\left(E\frac{\overline{\boldsymbol{v}_1}^T\boldsymbol{B}\boldsymbol{v}_1}{\overline{\boldsymbol{v}_1}^T\boldsymbol{A}\boldsymbol{v}_1} + \mu\frac{\overline{\boldsymbol{v}_2}^T\boldsymbol{B}\boldsymbol{v}_2}{\overline{\boldsymbol{v}_2}^T\boldsymbol{A}\boldsymbol{v}_2}\right), \; d_2 = \ldots$$

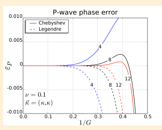


Fig 6: P-phase error: 2D elements of degree N=4,8,12

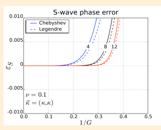


Fig 7: S-phase error: 2D elements of degree N=4,8,12

Case 3: 3D Acoustic

A mesh of cubes structured as in Case 2 yields

$$\boldsymbol{M}\ddot{\boldsymbol{u}}^* + c^2\boldsymbol{K}\boldsymbol{u}^* = \boldsymbol{0}$$

$$\boldsymbol{K}^e = \frac{h}{2}(\boldsymbol{A}\otimes\boldsymbol{A}\otimes\boldsymbol{B} + \boldsymbol{B}\otimes\boldsymbol{A}\otimes\boldsymbol{B} + \boldsymbol{B}\otimes\boldsymbol{A}\otimes\boldsymbol{A})$$

$$\boldsymbol{M}^e = \frac{h^3}{8}\boldsymbol{A}\otimes\boldsymbol{A}\otimes\boldsymbol{A}$$

$$\boldsymbol{u} \leftarrow e^{-i\omega^* t}e^{i(\kappa_1 e_1 + \ldots + \kappa_3 e_3)}\boldsymbol{v}_3\otimes\boldsymbol{v}_2\otimes\boldsymbol{v}_1$$

$$\omega^* = \frac{2c}{h}\sqrt{\frac{\overline{\boldsymbol{v}_1}^T\boldsymbol{B}\boldsymbol{v}_1}{\overline{\boldsymbol{v}_1}^T\boldsymbol{A}\boldsymbol{v}_1} + \frac{\overline{\boldsymbol{v}_2}^T\boldsymbol{B}\boldsymbol{v}_2}{\overline{\boldsymbol{v}_2}^T\boldsymbol{A}\boldsymbol{v}_2} + \frac{\overline{\boldsymbol{v}_3}^T\boldsymbol{B}\boldsymbol{v}_3}{\overline{\boldsymbol{v}_3}^T\boldsymbol{A}\boldsymbol{v}_3}}$$

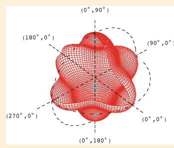


Fig 8: amplified dispersion / Chbyshev, N=2

Optimal Blending

Combination of consistent and lumped mass matrices [4,5]

$$\boldsymbol{A}^\vartheta = \vartheta\boldsymbol{A} + (1-\vartheta)\boldsymbol{A}^L, \quad 0 \le \vartheta \le 1,$$

Write 1D dispersion error as $\varepsilon_\vartheta(H)$, $H = 1/G$, and solve

$$\max_{0\le R\le 0.5}\left\{\min_{0\le\vartheta\le 1}\int_0^R \varepsilon_\vartheta(H)\,dH\;;\;|\varepsilon_\vartheta(H)| < tol \text{ in } [0,R]\right\}$$

for a prescribed tolerance tol

R_{max}: admissible domain of dispersion error

ϑ_{min}: optimal blending parameter

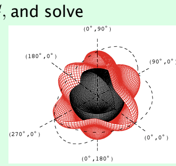


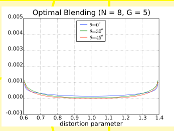
Fig 9: Optimal Blended x Consistent, N=2 [5]

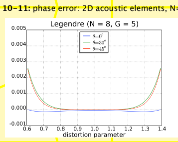
Dispersion by Mesh Distortion

Consider a periodical, non-rectangular mesh such that the first **M** elements define the mesh periodicity

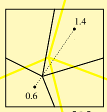
An element-by-element computation yields

$$\omega^* = c\sqrt{\frac{\sum_{e=1}^{M}\overline{\boldsymbol{w}^e}^T\boldsymbol{K}^e\boldsymbol{w}^e}{\sum_{e=1}^{M}\overline{\boldsymbol{w}^e}^T\boldsymbol{M}^e\boldsymbol{w}^e}}$$

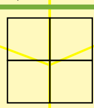


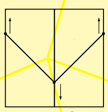


Figs 10–11: phase error: 2D acoustic elements, N=4

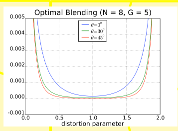


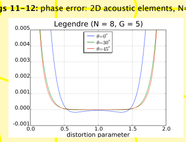
Test 1 [2]





Test 2





Figs 11–12: phase error: 2D acoustic elements, N=4

References

1 Belytschko & Mullen (1978) On dispersive properties of finite element solutions. In Miklowitz *et al*, *Modern problems in Elastic Wave Propagation*, Wiley
2 Cohen (2002) *Higher-Order Numerical Methods for Transient Wave Equations.* Springer
3 Komatitsch & Tromp (1999) Introduction to the spectral-element method for 3-D seismic wave propagation. *Geophys J Int* **139**:806-822
4 Marfurt (1990) Analysis of higher order finite-element methods. In Kelly & Marfurt, *Numerical modeling of seismic wave propagation*, SEG
5 Seriani & Oliveira (2007) Optimal blended spectral-element operators for acoustic wave modeling. *Geophysics* **72**(5):SM95-SM106