



CARDIFF
UNIVERSITY
PRIFYSGOL
CAERDYDD



School of Earth, Ocean & Planetary Sciences,
Cardiff University.

Civil & Computational Engineering Centre, School of
Engineering, Swansea University.



Adaptive Finite Element Methods For Geodynamics

D. Rhodri Davies^{1,2}

J. Huw Davies¹, Oubay Hassan², Ken Morgan², Perumal Nithiarasu².

¹ School of Earth, Ocean & Planetary Sciences, Cardiff University.

² Civil & Computational Engineering Centre, School of Engineering, University of Wales, Swansea.

DaviesDR3@cf.ac.uk
engdavdr@swan.ac.uk



Outline:

2-D

- Equations governing convection within Earth's mantle.
- Solution Strategy:
 - *Finite Element Method (FEM)*.
 - *Adaptive FEM (remeshing)*.
- Validation:
 - *Thermal convection*.
 - *Thermo-chemical convection*.
- Geodynamical Applications:
 - *Fluid Flow at a Mid-Ocean Ridge*.
 - *Subduction Zone Magmatism*.
- 2-D Conclusions.

3-D

- Grid adaptivity in 3-D.
- Methods utilized.
 - *Adaptive Multigrid*.
- Implementation.
 - *TERRA*.
- Validation.
 - *Finite Amplitude Convection*.
 - *Stokes Flow*.



The Equations Governing Mantle Convection:

Earth's mantle is solid. However it deforms slowly over time (mantle convection) and consequently, in numerical models it is treated as a **highly viscous fluid**.

The essential equations governing the motion of a viscous fluid are (**Stokes flow**):

Conservation of **Mass** (Continuity):

$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

Conservation of **Momentum**:

$$\nabla^2 \mathbf{u} - \nabla p = Ra T \hat{k} \quad Ra = \frac{\beta \rho g \Delta T d^3}{\kappa \mu} \quad (2)$$

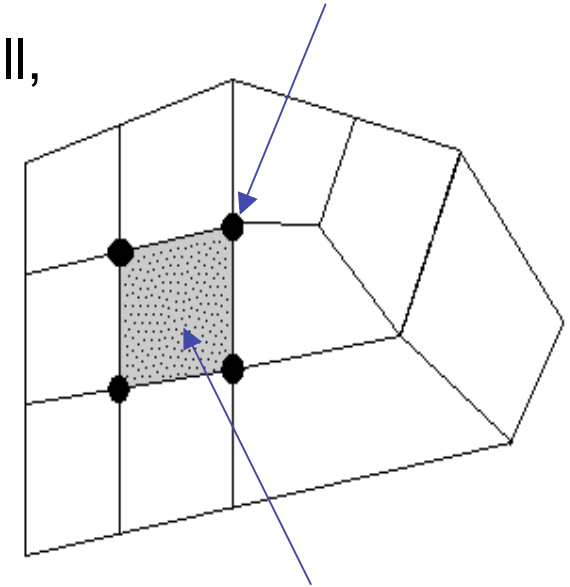
Conservation of **Energy**:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T \quad (3)$$

The Finite Element Method:

Node

- The computational domain comprises many small, interconnected, sub-regions / *elements*.
- Complex PDE's are reduced to either linear or non-linear simultaneous equations.
- The FE discretization procedure reduces the continuum problem, which has an infinite number of unknowns, to one with a finite number of unknowns at specified points referred to as *nodes*.
- Since the FEM is based upon an integral formulation it is readily implemented on arbitrary discretizations, i.e. unstructured grids.



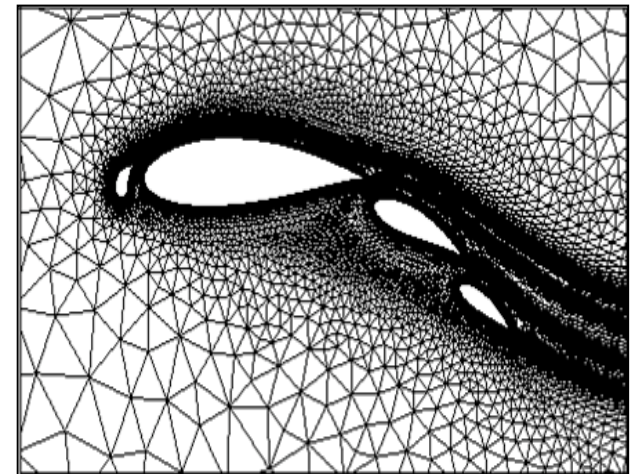
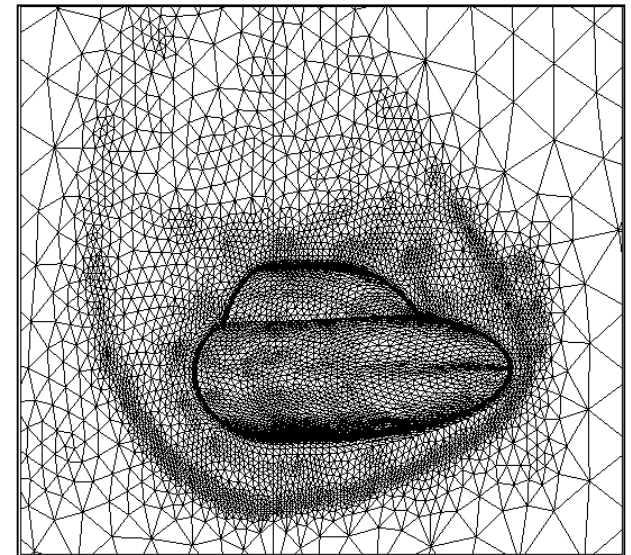
Element

Small elements → High resolution. Also, greater computational expenditure!



Adaptive Finite Element Methods:

- The finite element solution provides an **approximation** to the exact solution. However, even on very fine grids, there remains an **error** in this solution. Adaptive mesh refinement is one of several techniques that provide a means to reduce this error.
- *Localized flow features can be resolved in the most efficient and cost-effective manner.*
- The **two** main ingredients are:
 - *Error indication capability, which will dictate the required grid resolution.*
 - *A procedure that is capable of **constructing a grid** with the desired characteristics.*





Adaptive Finite Element Methods (Continued):

- Adaptive procedures fall into **two** categories:

- *h*-refinement - the same class of elements continue to be used but are changed in size, in some locations made larger, and in others made smaller, to provide maximum economy in reaching the desired solution.

Mesh regeneration / 'remeshing'.

Element Subdivision.

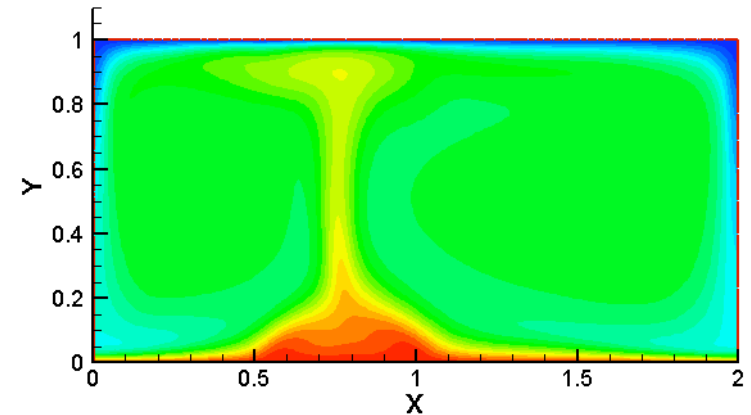
r-refinement.

- *p*-refinement - the same element size is utilized, but the order of the polynomial is increased / decreased as required.



‘ConMan’:

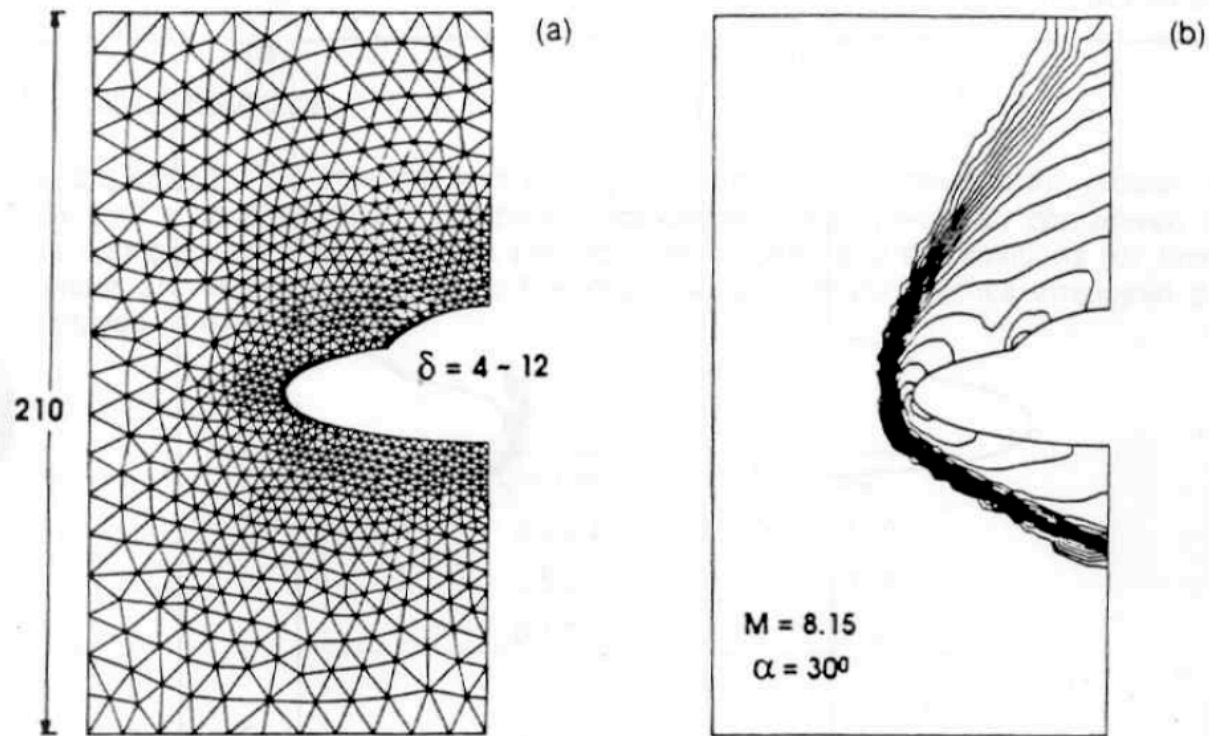
- Written by Scott King (King *et al.* 1990).
- 2-D, incompressible, infinite Prandtl Number finite element code.
- Equations (1) and (2) are solved by the well-known **penalty method**.
- **Bilinear, quadrilateral** elements.
- Equation (3) solved using the streamline-upwind Petrov-Galerkin (**SUPG**) method.
- 2nd order **predictor-corrector** algorithm employed for time marching.



Adaptive Remeshing in 2-D.....

Adaptive Remeshing Strategy for Steady-State Simulations:

The problem is *solved initially* on a grid fine enough to capture the basic physics of the flow.



Mesh 1 – 1599 Elements, 882 Nodes.

Remeshing then involves the following stages:



Adaptive Remeshing Strategy – Stage 1 (1)

The solution is analyzed through some kind of *error indication procedure*, to determine locations where the mesh fails to provide an adequate definition of the problem.

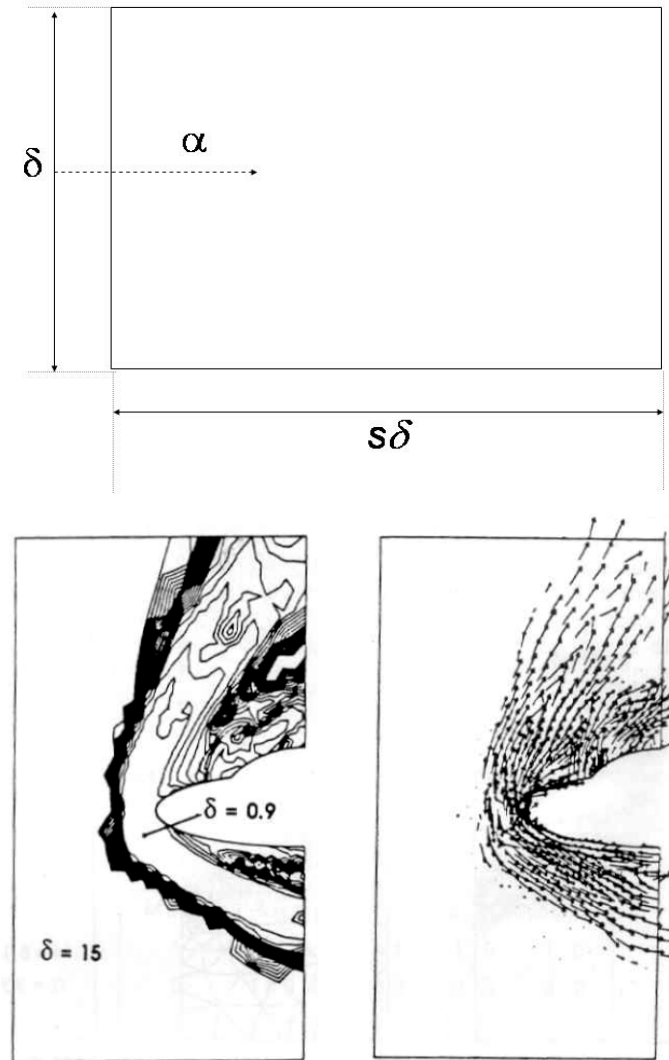
Additional grid points needed?

Too many grid points at certain locations within the domain?

An *interpolation based local error indicator* is employed in this study, based upon nodal solution curvatures (2nd derivatives).

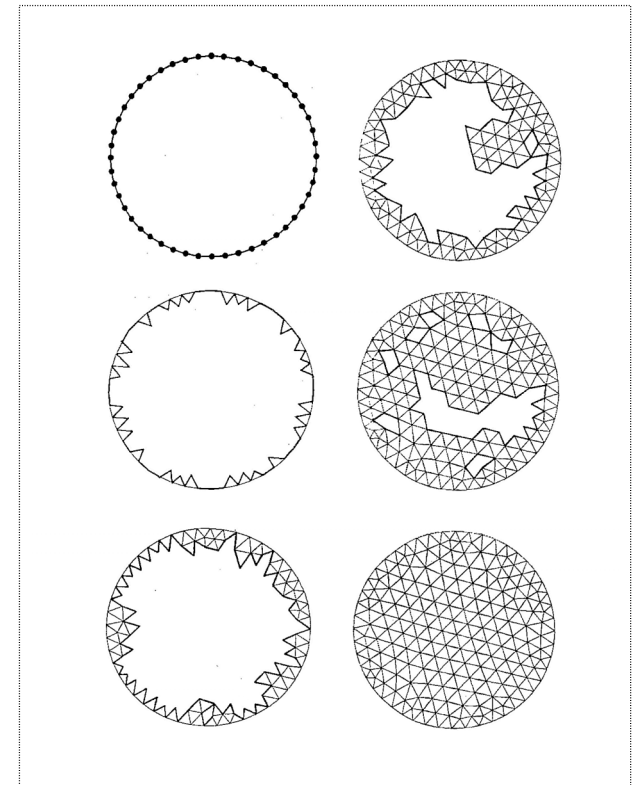
Adaptive Remeshing Strategy – Stage 1 (2)

- The computed solution is used to determine 'optimum' nodal values for α , δ and S .
- A certain 'key' variable must be identified and then the error indication process can be performed in terms of this variable.
- Examples include:
 1. Temperature.
 2. Velocity
 3. Density.



Adaptive Remeshing Strategy – Stage 2 (1)

- The information yielded by this error indication process is utilized by an *automatic mesh generator* to produce an improved mesh.
- A variant of the so-called *advancing front technique* is used to regenerate the meshes in this study (Peraire *et al.* 1987).
- Capable of generating meshes that conform to an externally prescribed spatial distribution of element size.

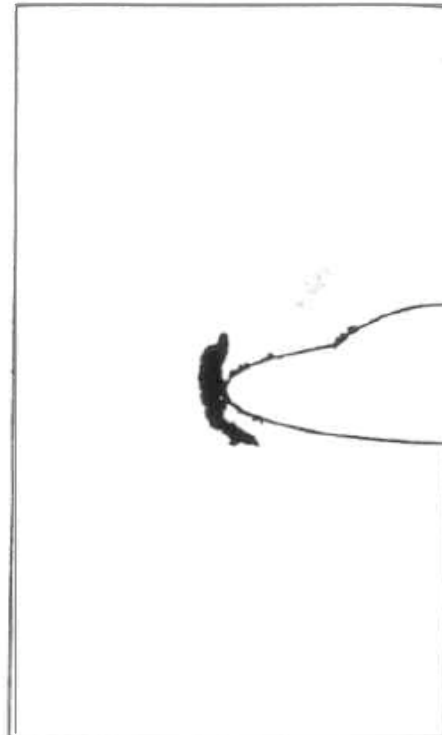


Adaptive Remeshing Strategy – Stage 2 (2)

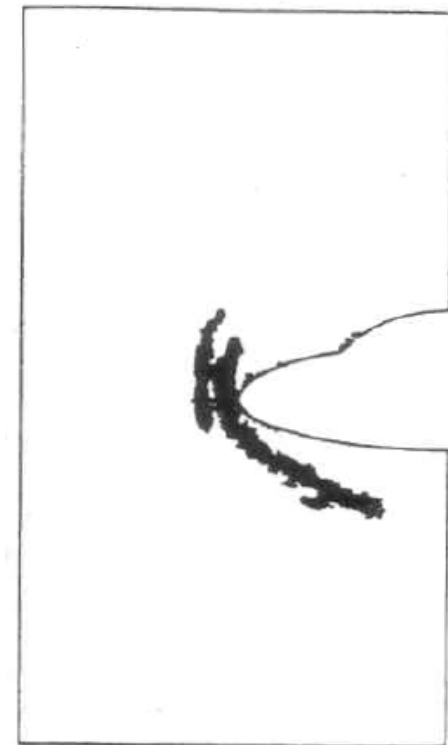
Illustration of the regeneration procedure:



(a)



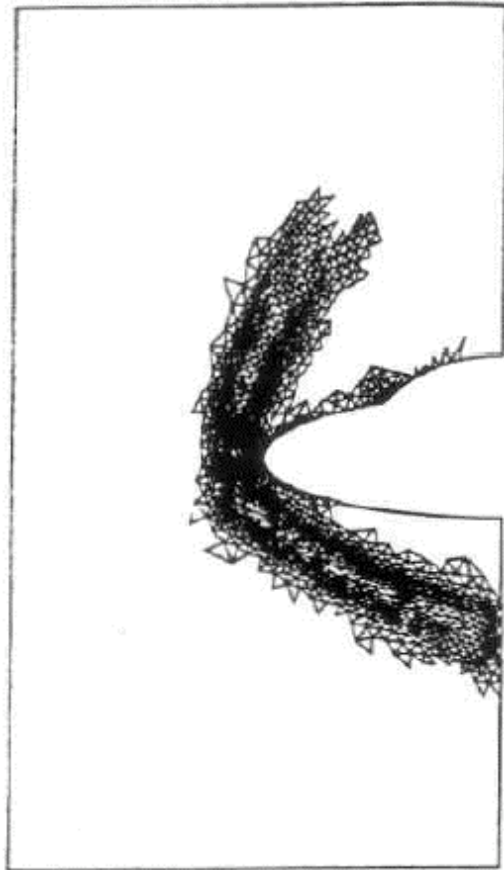
(b)



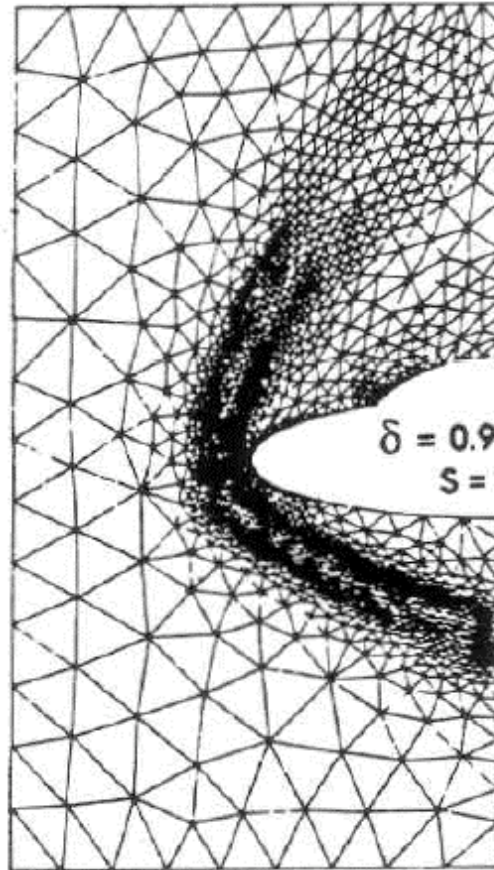
(c)

Different stages during the regeneration process.

Adaptive Remeshing Strategy – Stage 2 (3)



(d)

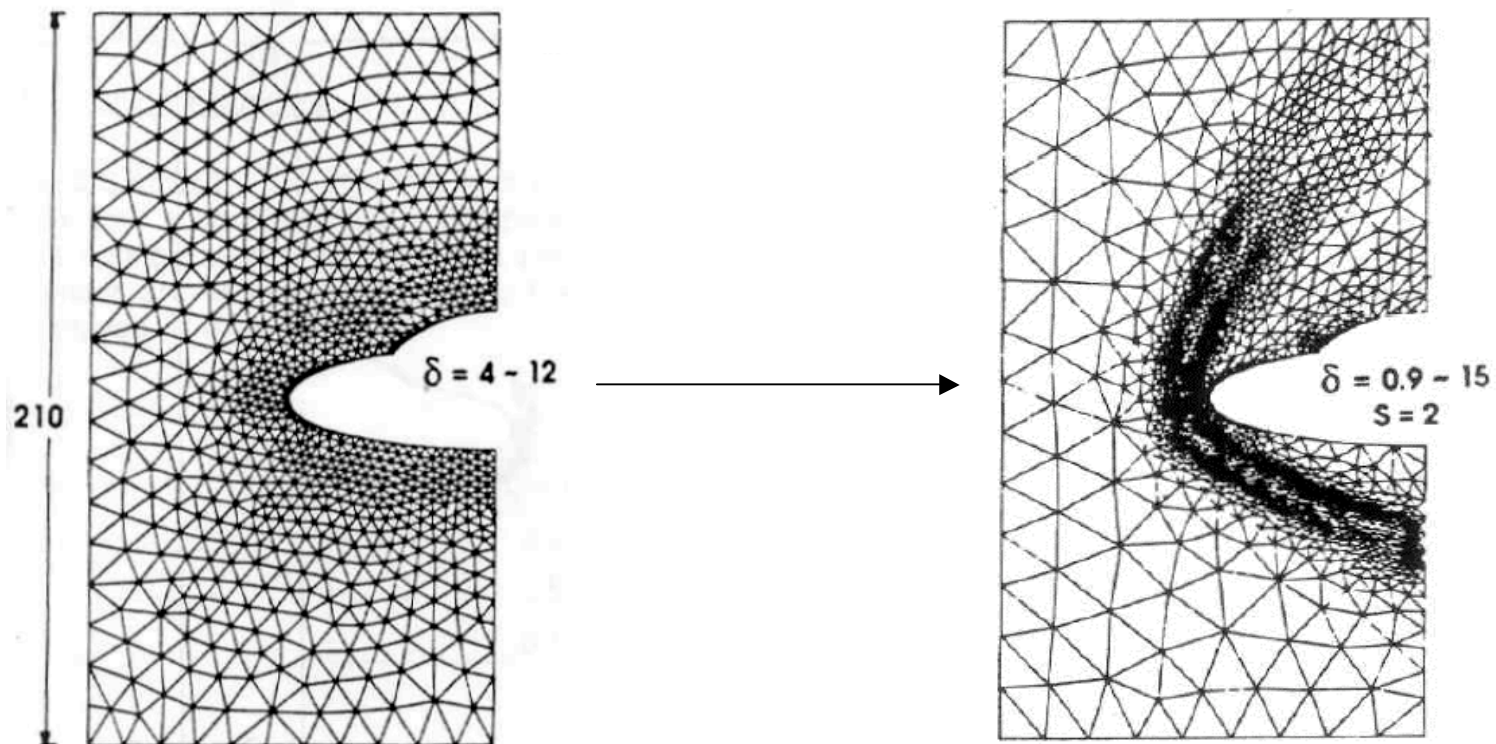


(e)

Final
regenerated
mesh.

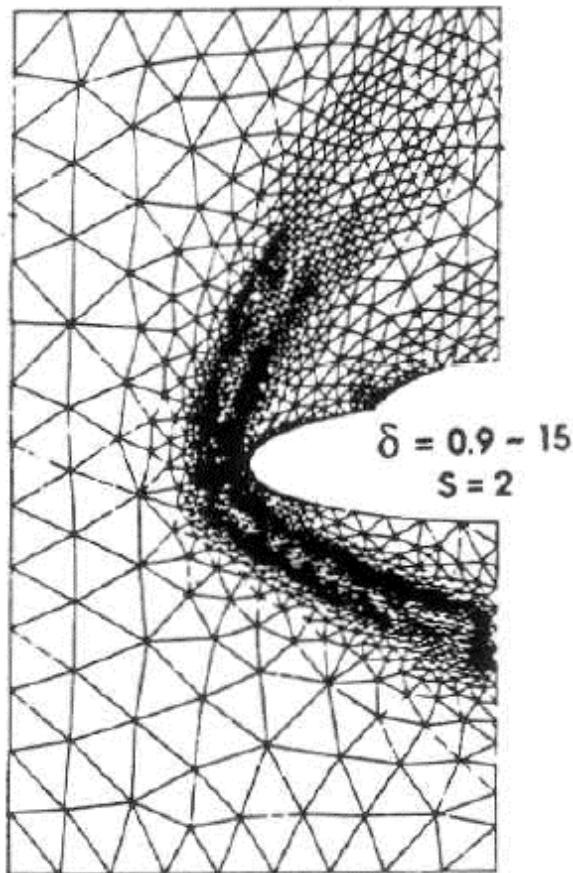
Adaptive Remeshing Strategy – Stage 3:

The original solution is *interpolated* between meshes



Adaptive Remeshing Strategy – Stage 4:

The solution procedure then *continues on the new mesh*.



Mesh

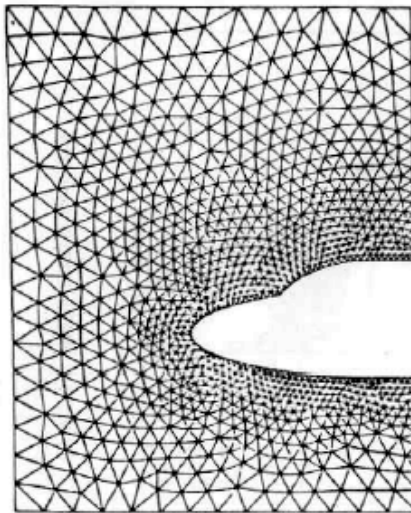


Mach number solution

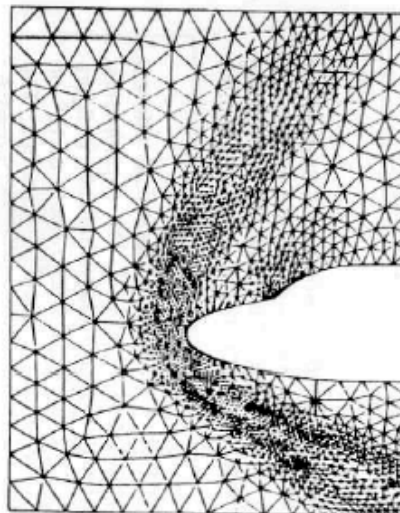


Original solution

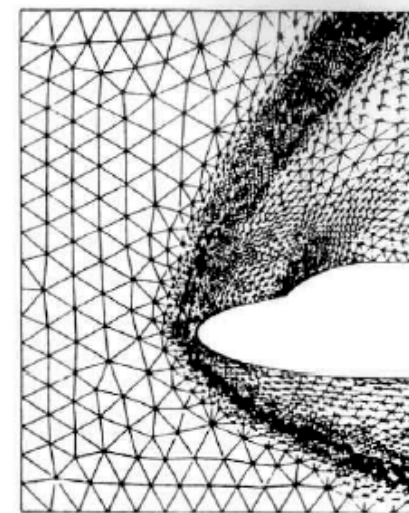
→ Repeated until the desired level of accuracy has been achieved.



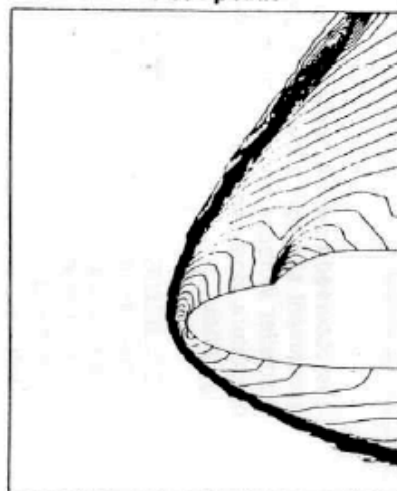
FIRST MESH
2 027 elements
1 110 points



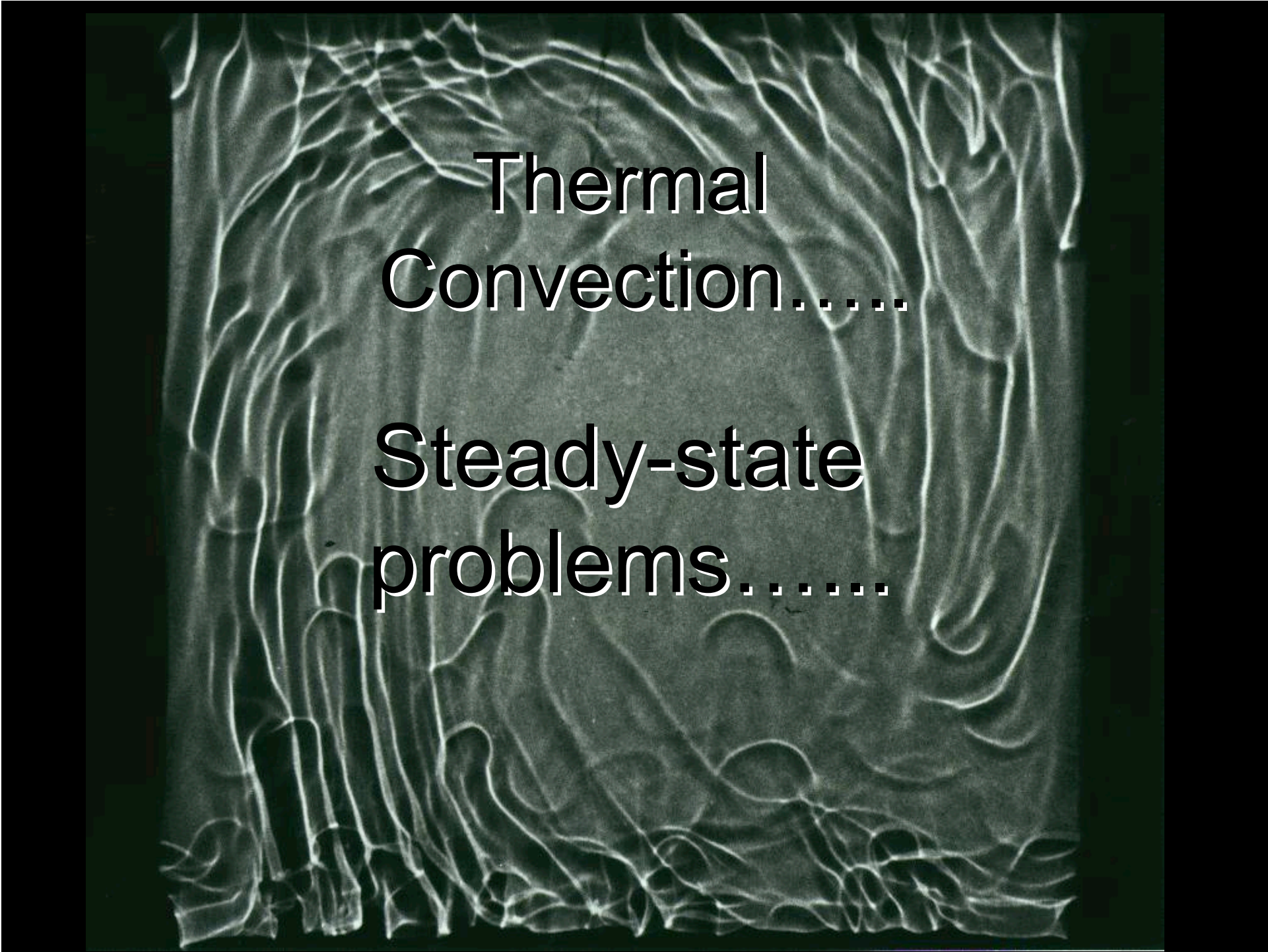
SECOND MESH
3 557 elements
1 864 points



THIRD MESH
6 403 elements
3 294 points



Geodynamic Benchmarks.....

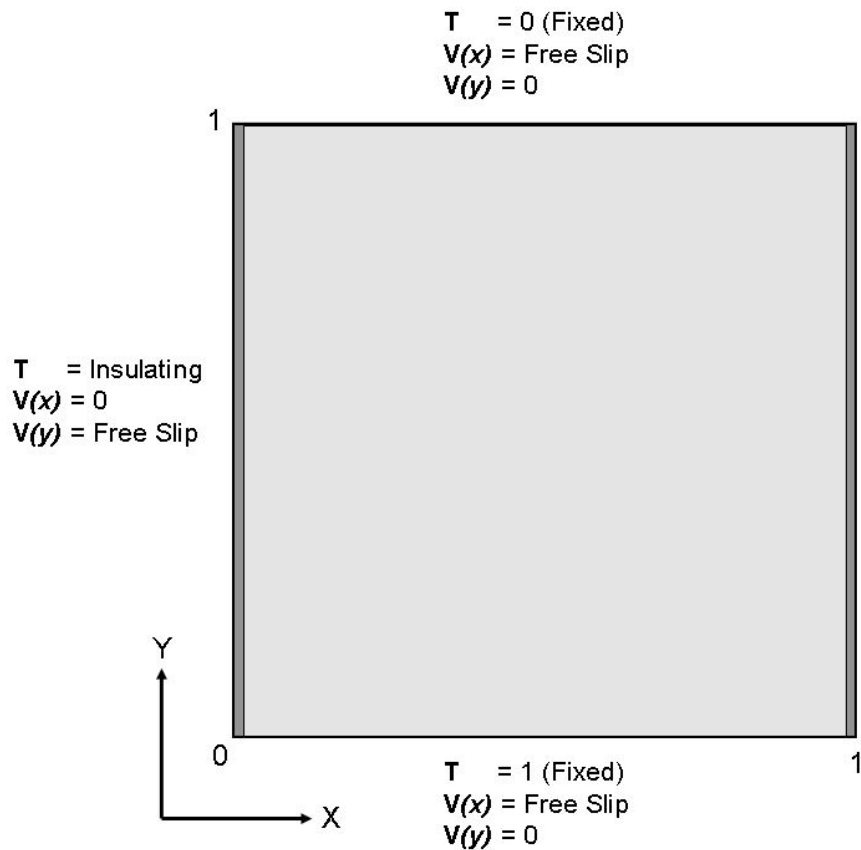


**Thermal
Convection.....**

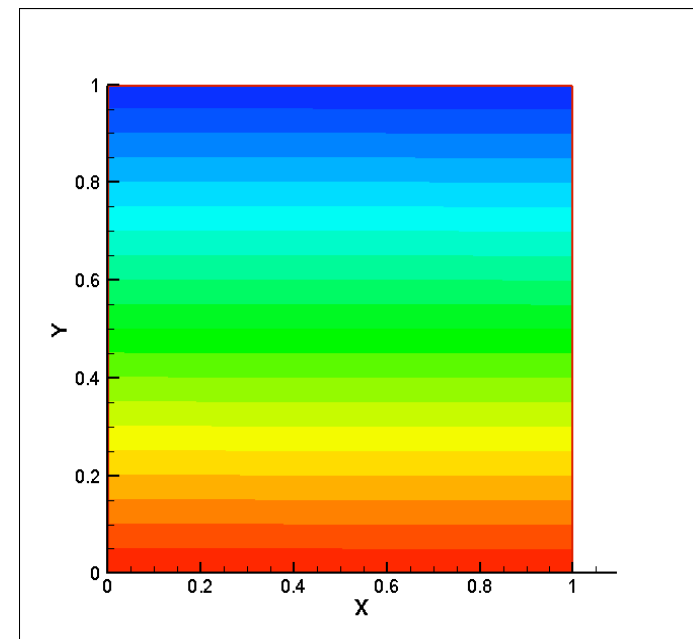
**Steady-state
problems.....**

Problem Description:

Boundary Conditions:



Initial Condition: Perturbed conductive profile.



Isoviscous, Isochemical media.



Data Calculated:

The Nusselt Number:

$$Nu = \int_0^l -\frac{\partial T}{\partial y}(x, y = 1) dx$$

The RMS velocity:

$$V_{RMS} = \sqrt{\frac{1}{V} \int_V \|\mathbf{u}\|^2}$$

Temperature gradients at domain corners.

$$q = -\frac{\partial T}{\partial y} \quad \begin{array}{l} q_1 \rightarrow x = 0, y = 1, \\ q_2 \rightarrow x = 1, y = 1, \end{array}$$

Solution Error: Calculate % error for each output and subsequently take the mean of these four percentages.

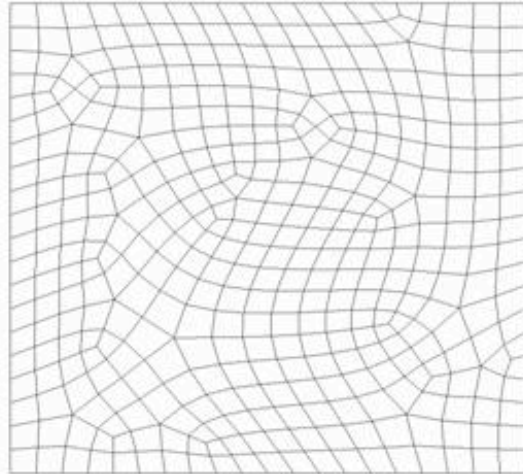
Results compared with benchmark solutions of Blankenbach *et al.* (1989).



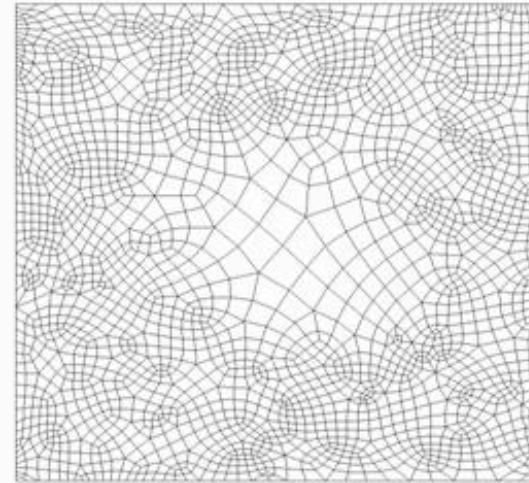
Remeshing Strategy:

- A **steady-state** solution is achieved here.
- Remeshing is performed when the solution converges to a steady state on a given grid.
- The process is terminated when an optimal mesh has been produced, i.e. the solution does not improve with the remeshing procedure.
- The error indicator employed in this case is based upon **nodal temperature curvatures**.

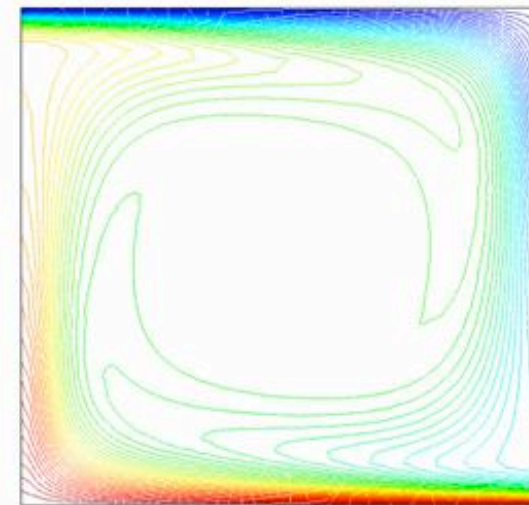
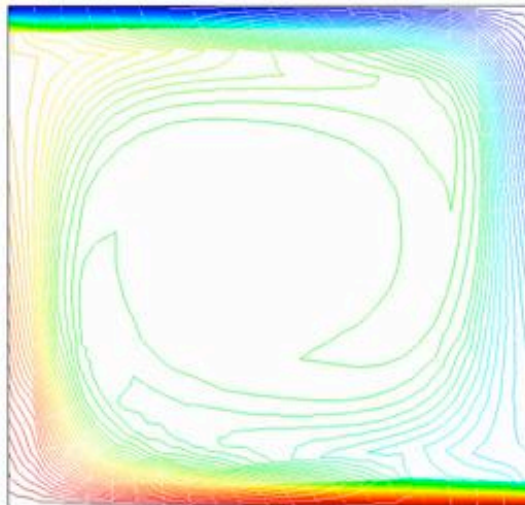
Results – Thermal Convection (1):



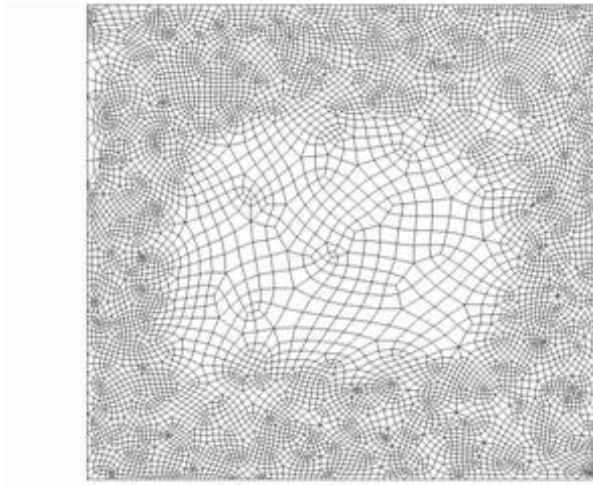
Mesh 1 – 453 Nodes, 412 Elements.



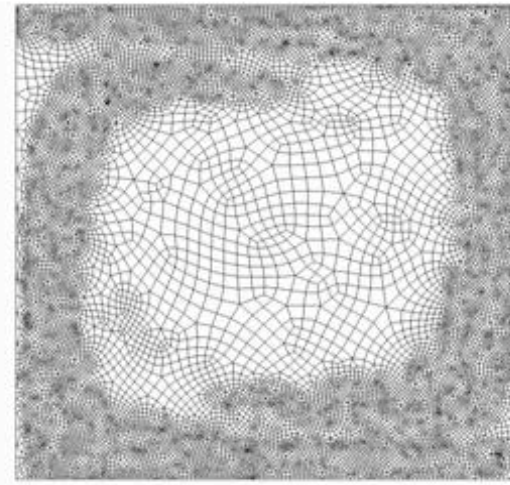
Mesh 2 – 2493 Nodes, 2390 Elements.



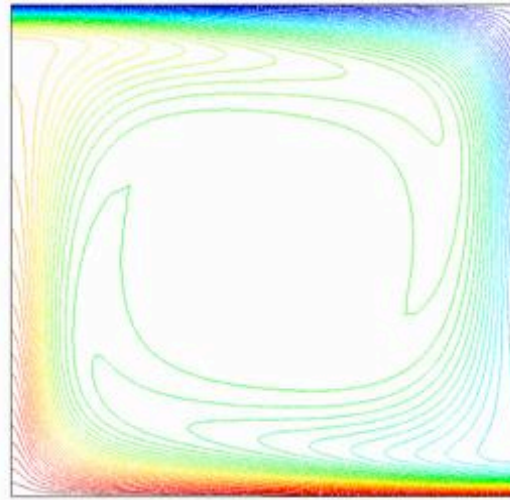
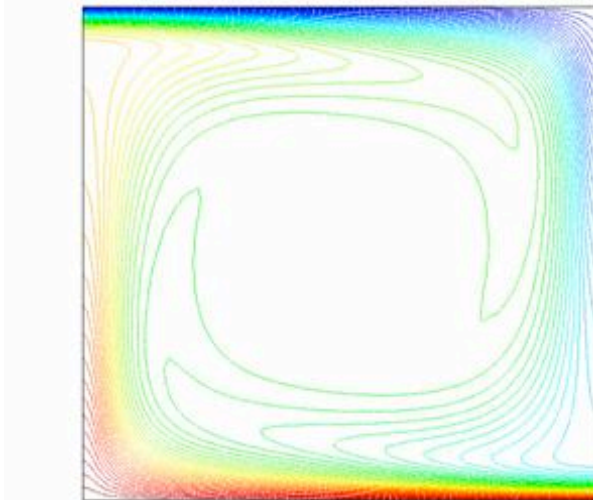
Results – Thermal Convection (2):



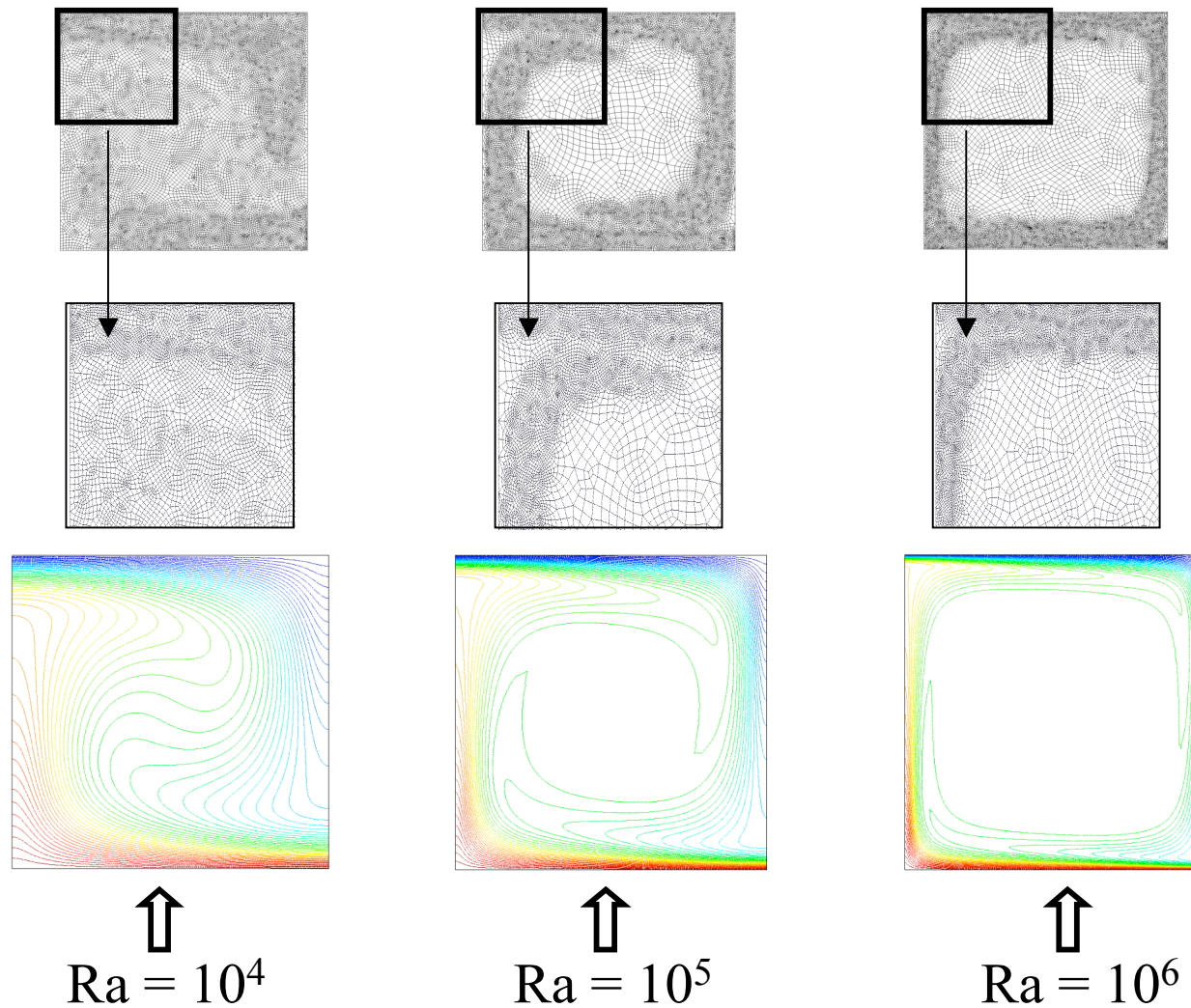
Mesh 3 – 7469 Nodes, 7321 Elements.



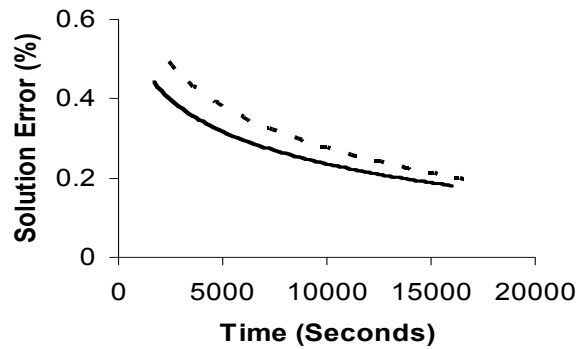
Mesh 4 – 15961 Nodes, 15722 Elements.



Results – Thermal Convection (3):

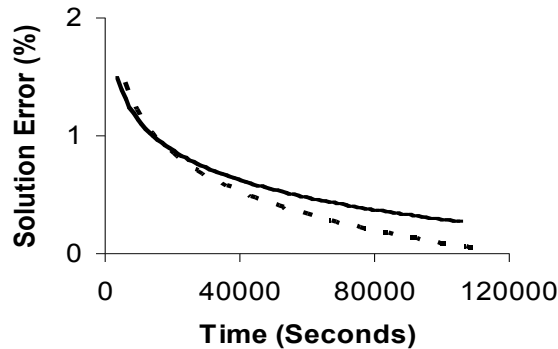


Results – Thermal Convection (4) – Processing Efficiency:

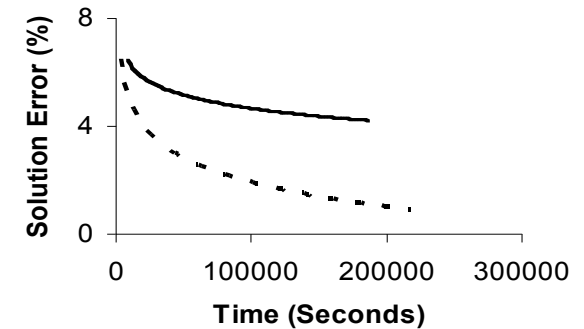


$Ra = 10^4$

Adapted
Uniform



$Ra = 10^5$



$Ra = 10^6$

Adaptive, unstructured meshes are more accurate and computationally more efficient than uniform, structured meshes.



Thermo-Chemical
Convection.....

Evolving problems.....



Representing the Chemical Field:

1. *Tracer Particle Methods* – tracers are distributed throughout the computational domain with different tracers representing differing values of chemical density.
2. *Marker Chain Methods* – form of boundary integral method where only interfaces between layers of differing densities are discretized.
3. *Field Approach / Grid-Based Methods* – chemical density is specified as a continuous field, similar to the temperature.

$$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C = \frac{1}{Le} \nabla^2 C$$

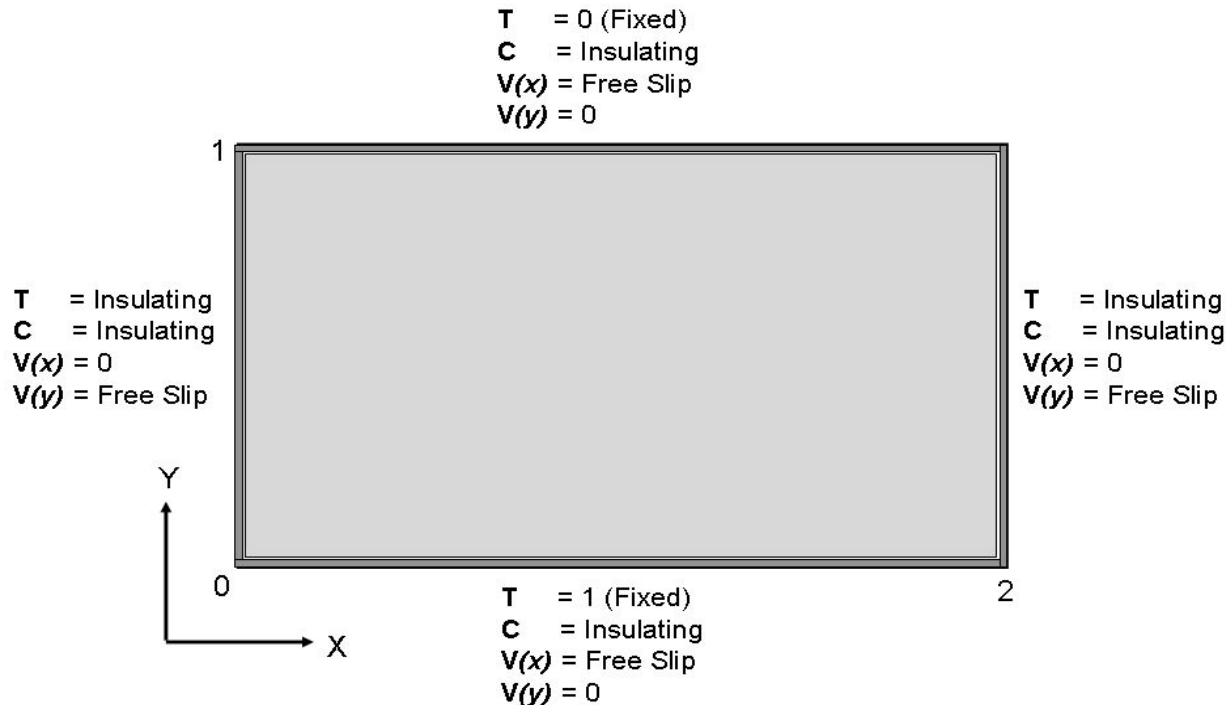
A filtering scheme (Lenardic & Kaula, 1997) is employed to remove spurious numerical undershoot and overshoot features.



Grid Based Methods:

- Suffer from numerical diffusion, leading to greater entrainment rates in numerical simulations (van Keken *et al.* 1997).
- This numerical diffusion is predominantly caused by insufficient resolution, a factor that is naturally addressed by the AFEM.
- Two hypotheses are tested:
 1. *The greater resolution endorsed by the AFEM will reduce artificial diffusion.*
 2. *This reduced diffusion, in turn, will see grid based methods yielding results that are consistent with those achieved using tracer particle and marker chain methods.*

Problem Description:

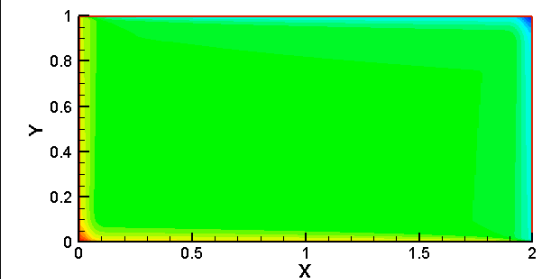


Isoviscous, $1/Le = 10^{-6}$, $Ra = 3 \times 10^5$, $Rb = 4.5 \times 10^5$

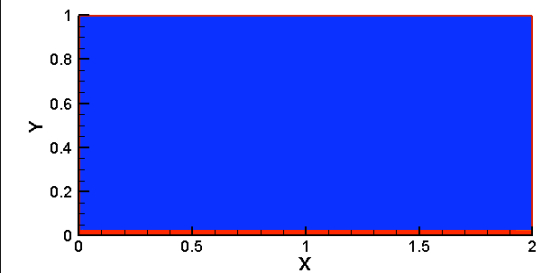
Data Calculated:
Relative Entrainment

$$e = \frac{1}{\lambda d_b} \int_{d_e}^l C dV$$

Analytical expression of temperature based on boundary layer theory.

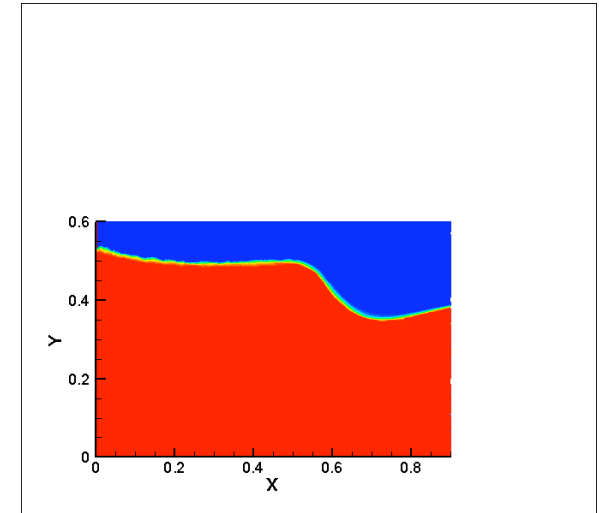
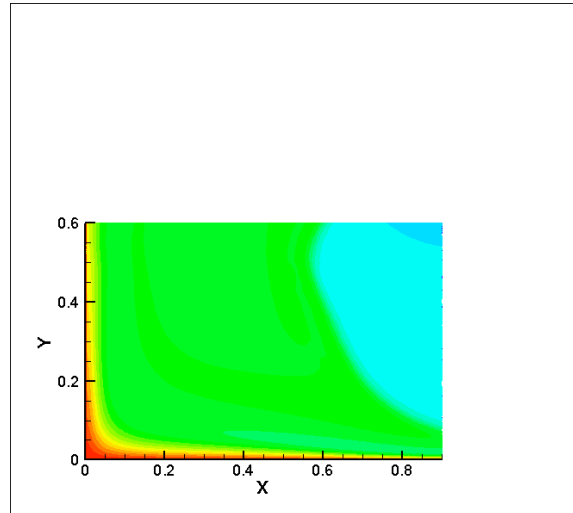
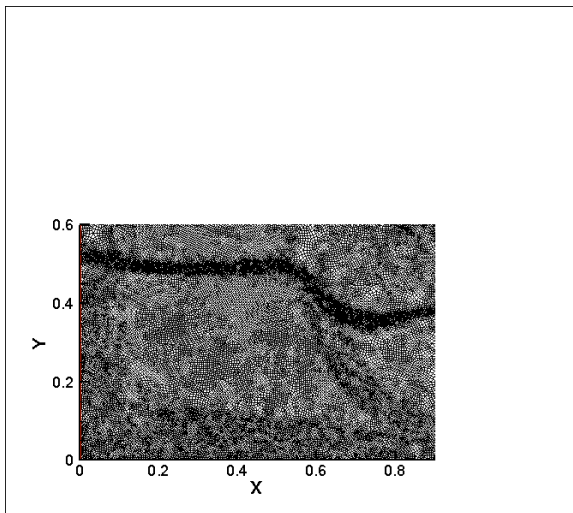


Deep-seated, thin dense layer (0.025)



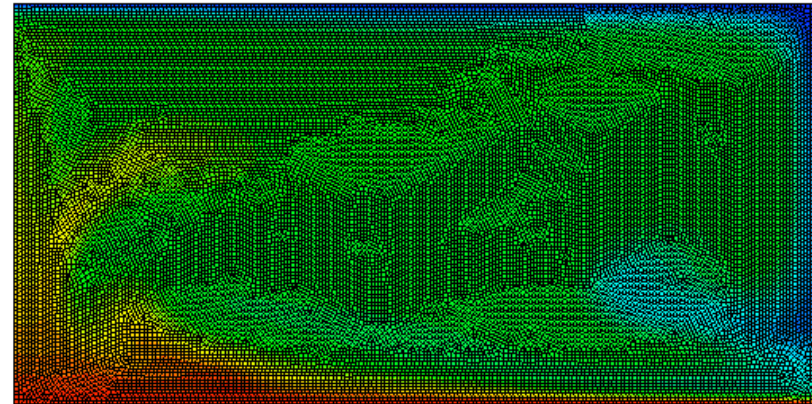
Remeshing Strategy:

- Remeshing frequency of 2000 time steps.
- The error indicator employed is based upon a combination of temperature and composition, as opposed to temperature alone - nodal solution derivatives are calculated for both T and C .
- The largest values yielded are then selected as derivatives for that particular node.

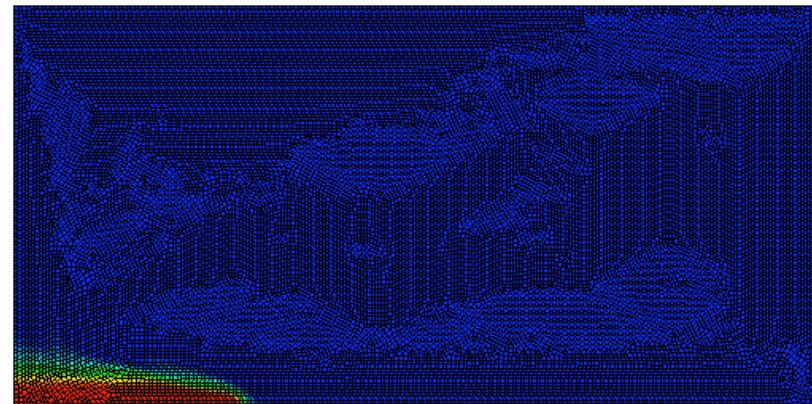


Results - Animations:

Temperature Field

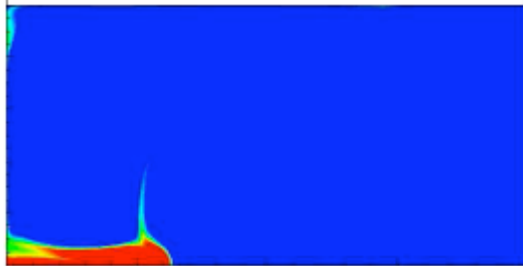


Compositional Field

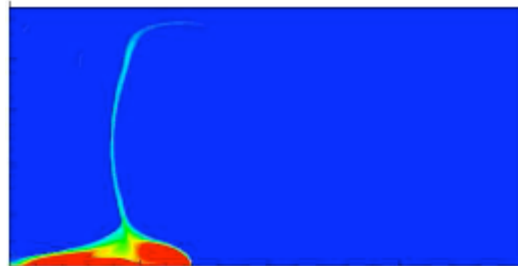


Results – Uniform Grid:

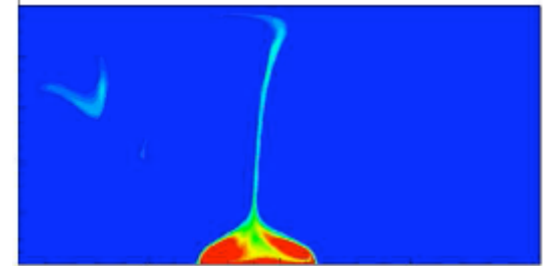
T = 0.01



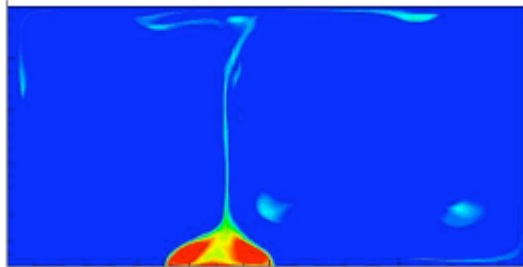
T = 0.02



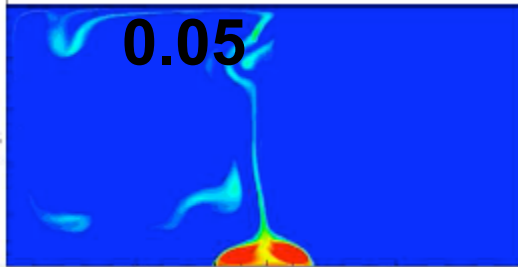
T = 0.03



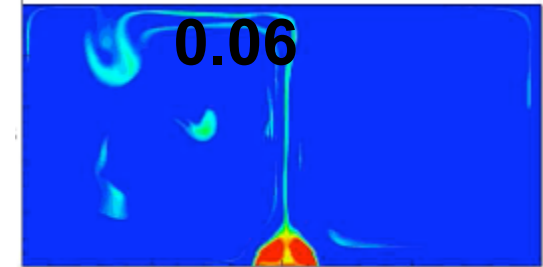
T = 0.04



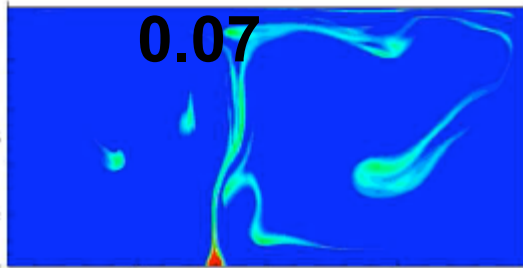
**T =
0.05**



**T =
0.06**



**T =
0.07**

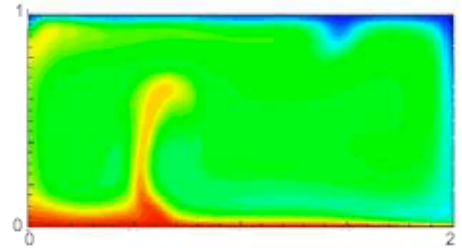
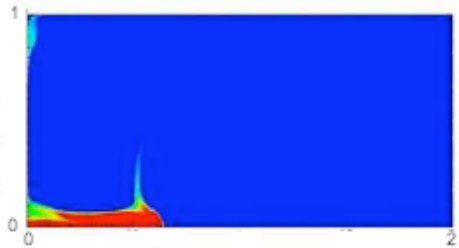


Diffusion dominates and, by $t = 0.07$, the dense layer has virtually disappeared.

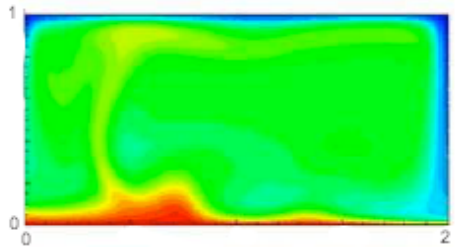
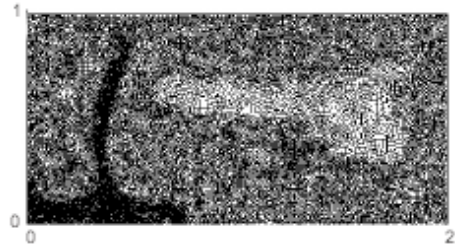
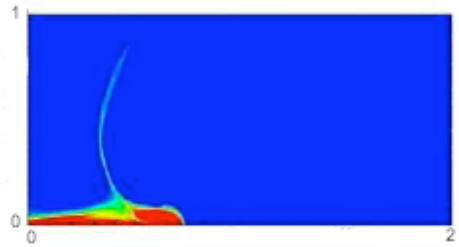


Results – Adapted Grids (1):

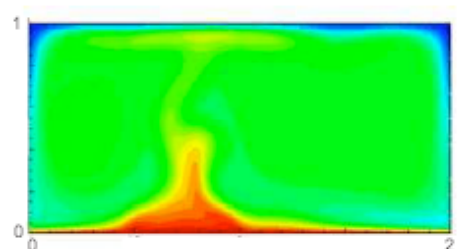
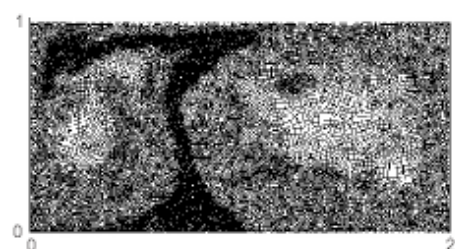
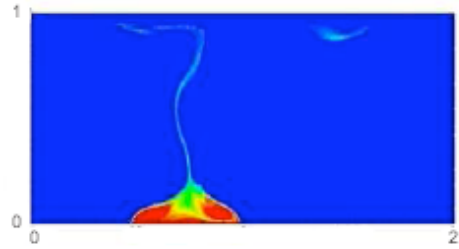
T = 0.01



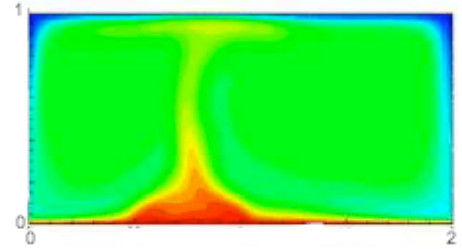
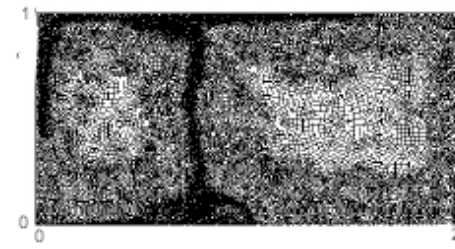
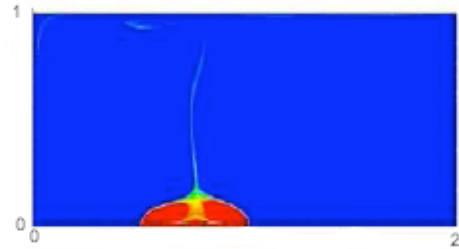
T = 0.02



T = 0.03

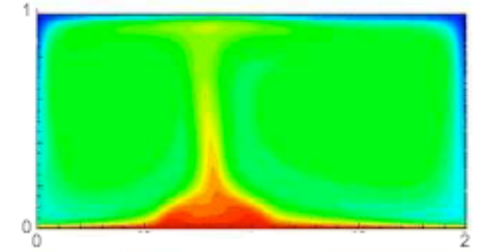
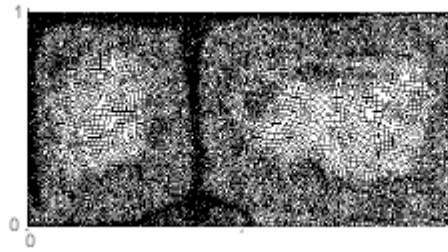
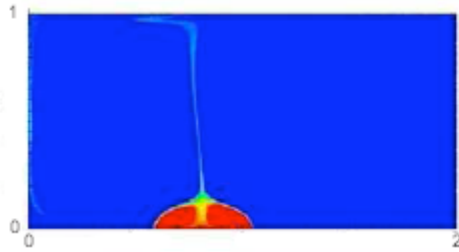


T = 0.04

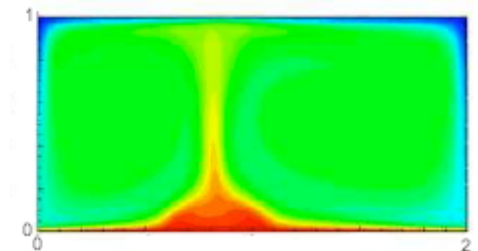
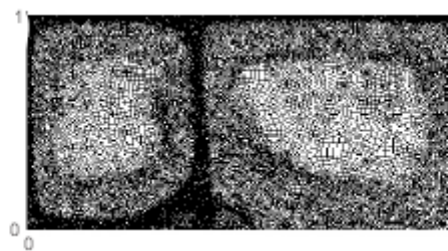
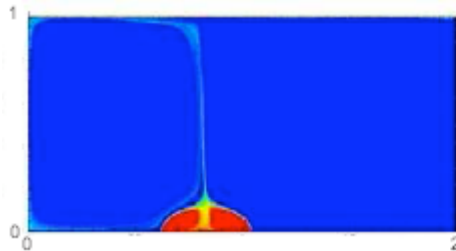


Results – Adapted Grids (2):

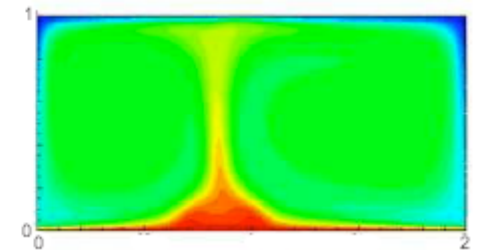
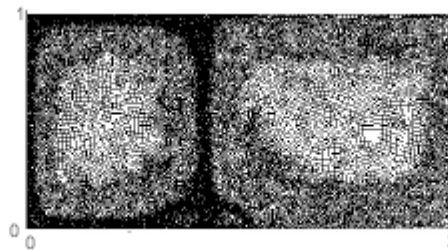
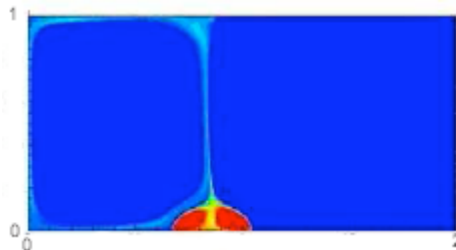
T = 0.05



T = 0.06

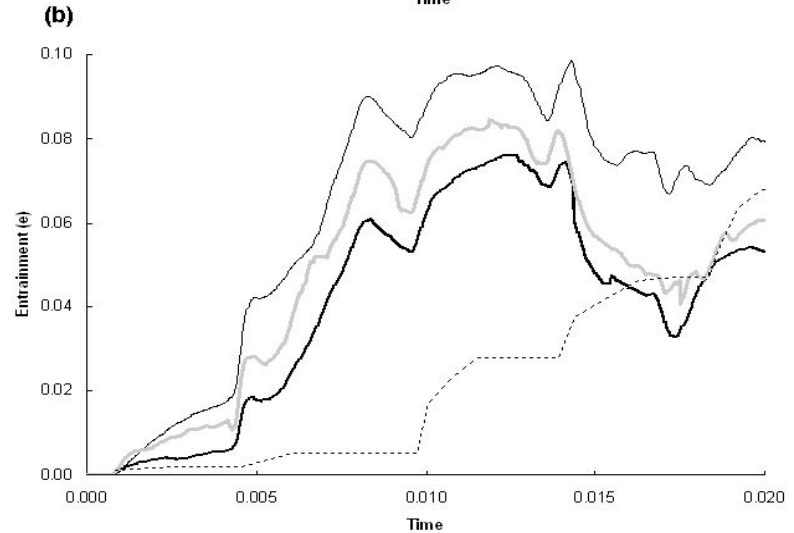
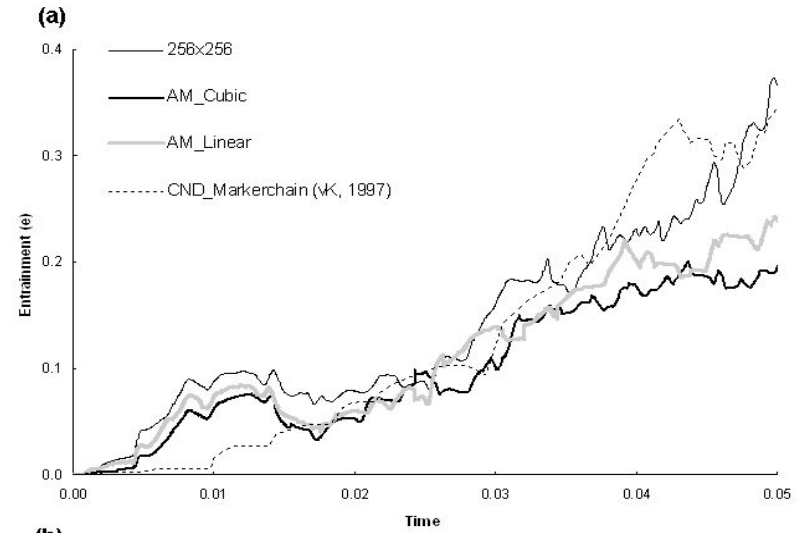
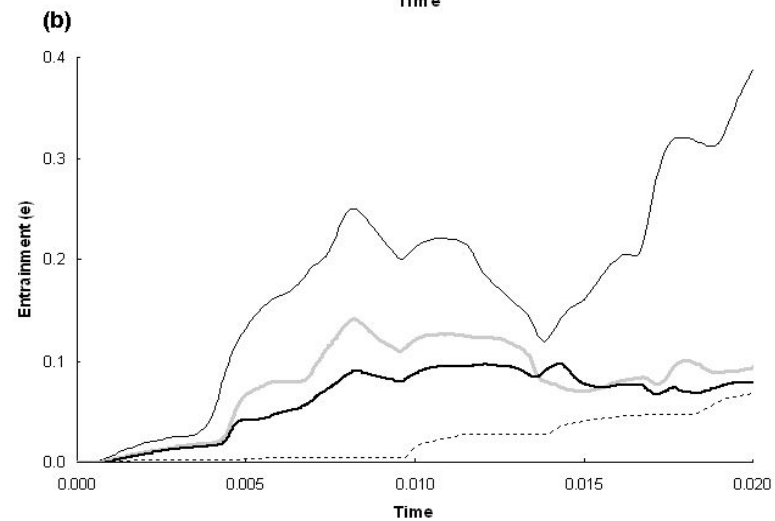
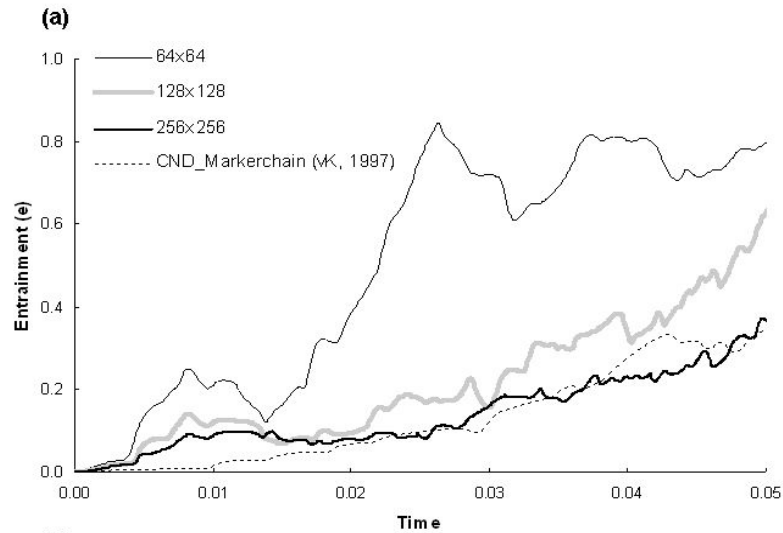


T = 0.07



Dense pile remains extremely coherent. More consistent with previous results from marker chain and tracer particle methods.

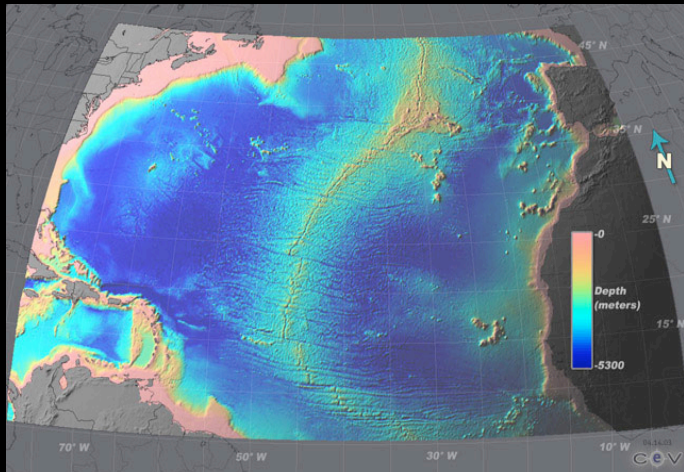
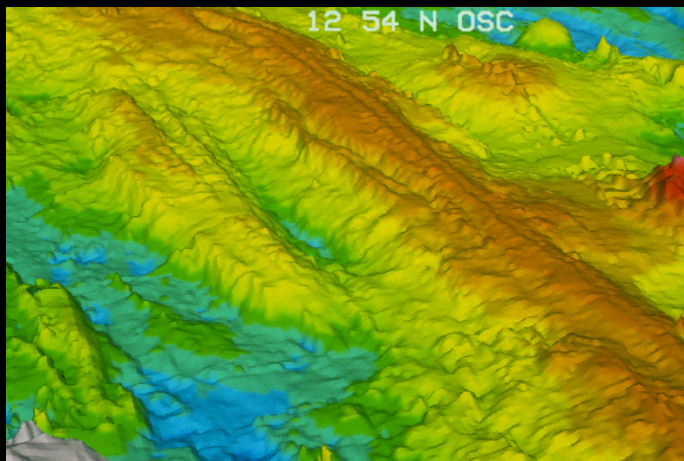
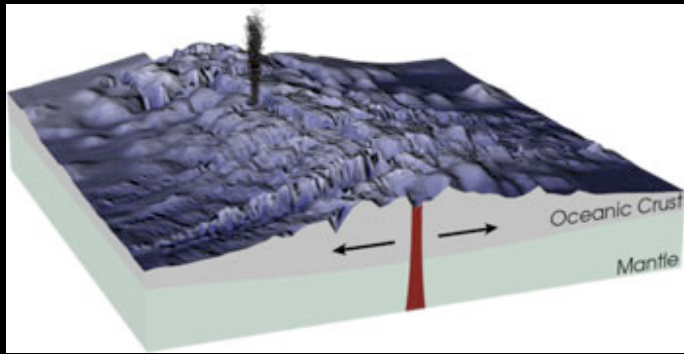
Relative Entrainment Against Time:





Geodynamical Applications:

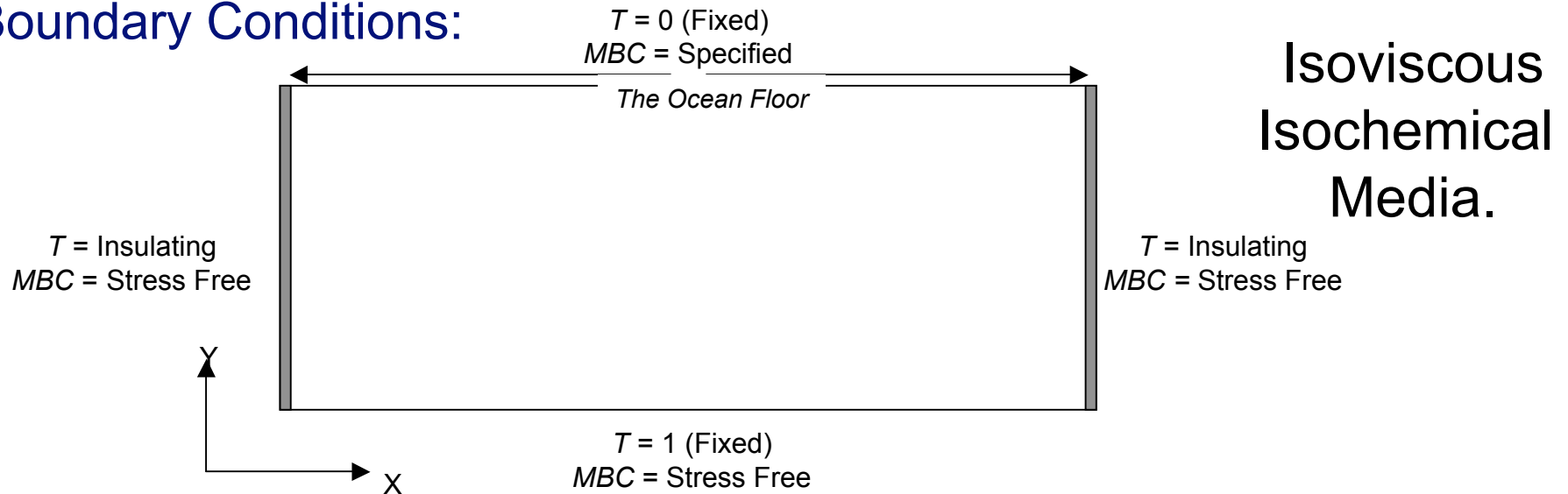
<http://www.mondolithic.com/images>



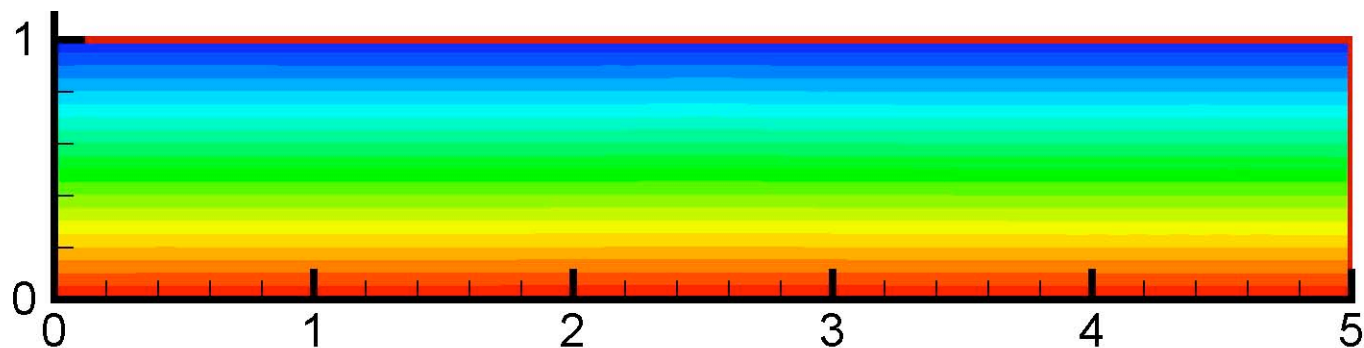
Mid-Ocean Ridges:

Problem Description:

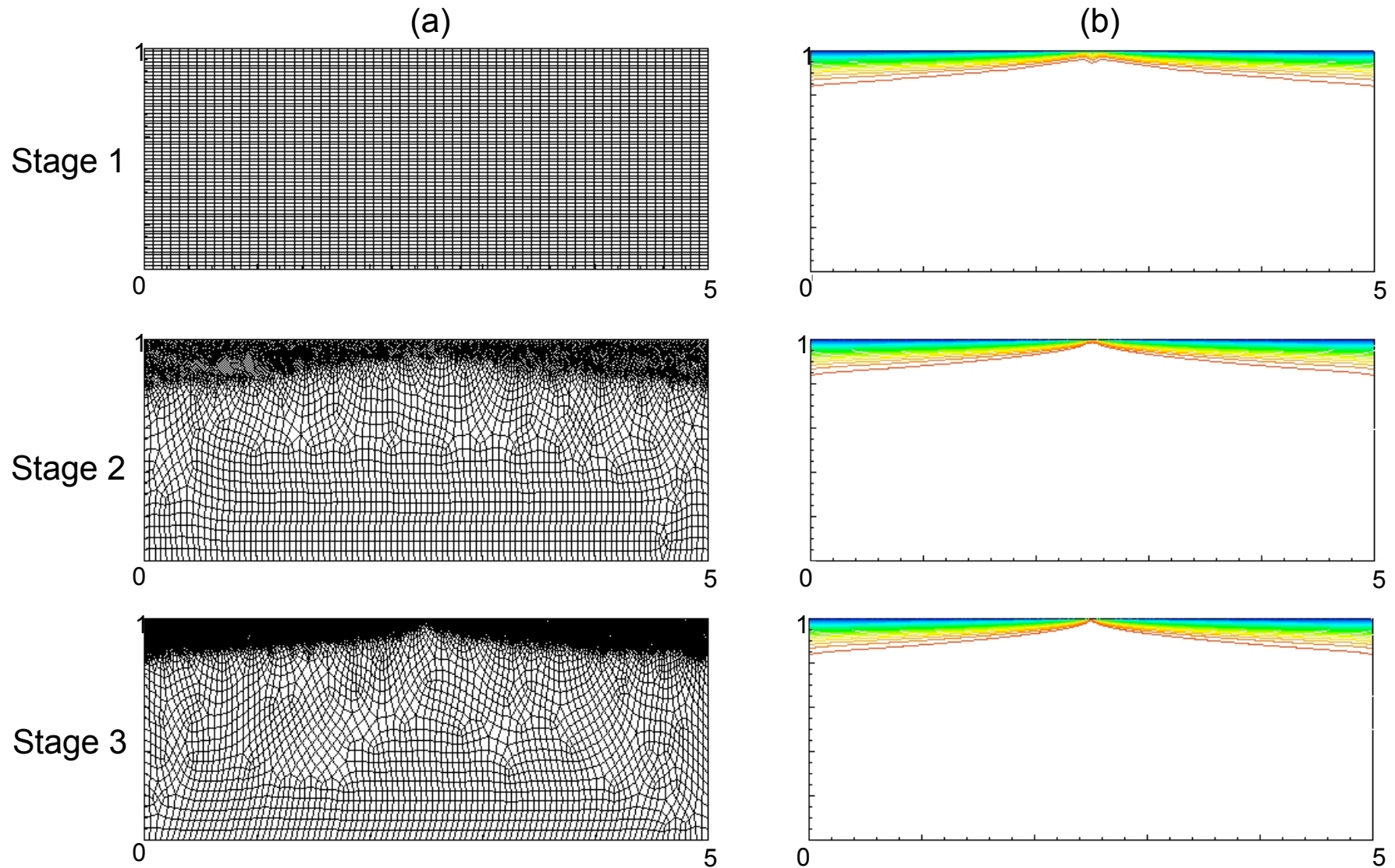
Boundary Conditions:



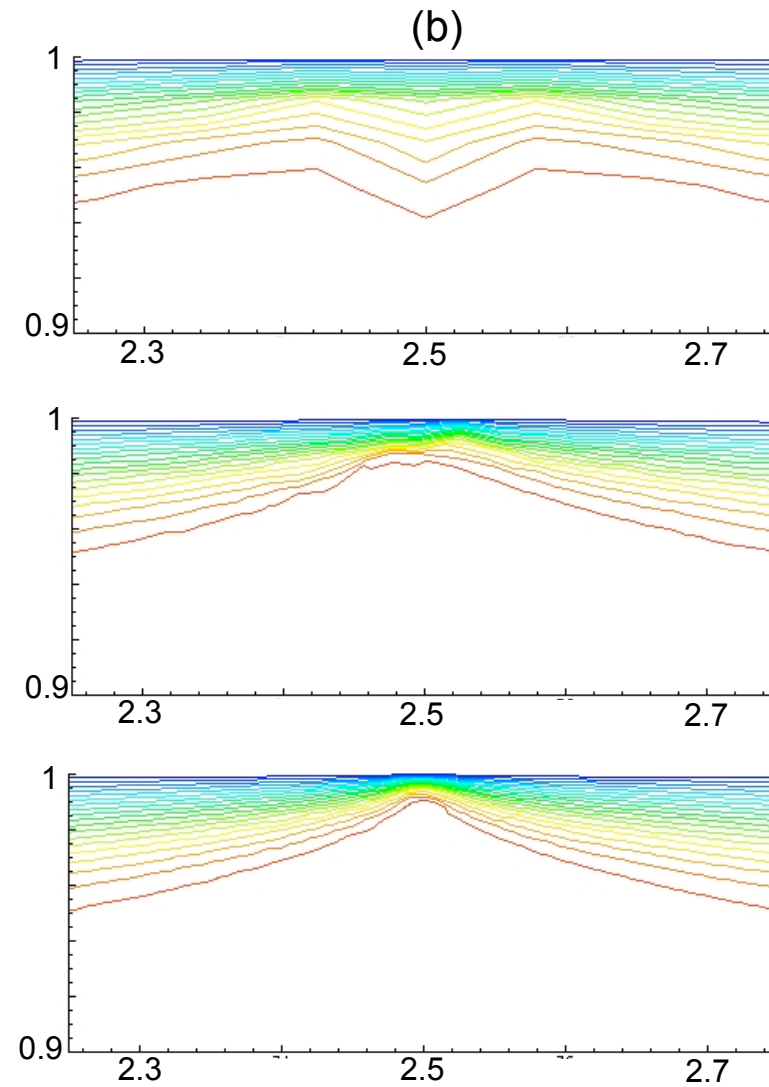
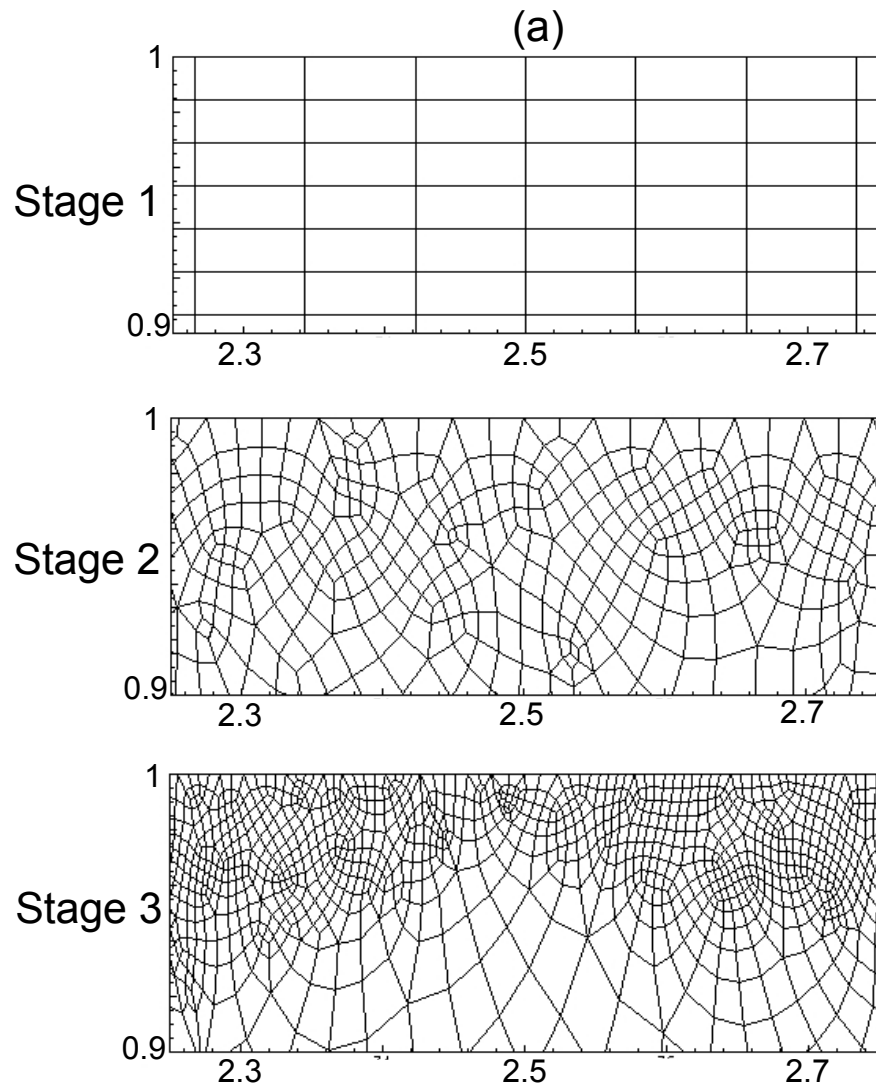
Initial Temperature Profile:



Mid Ocean Ridge-Simulations (1):

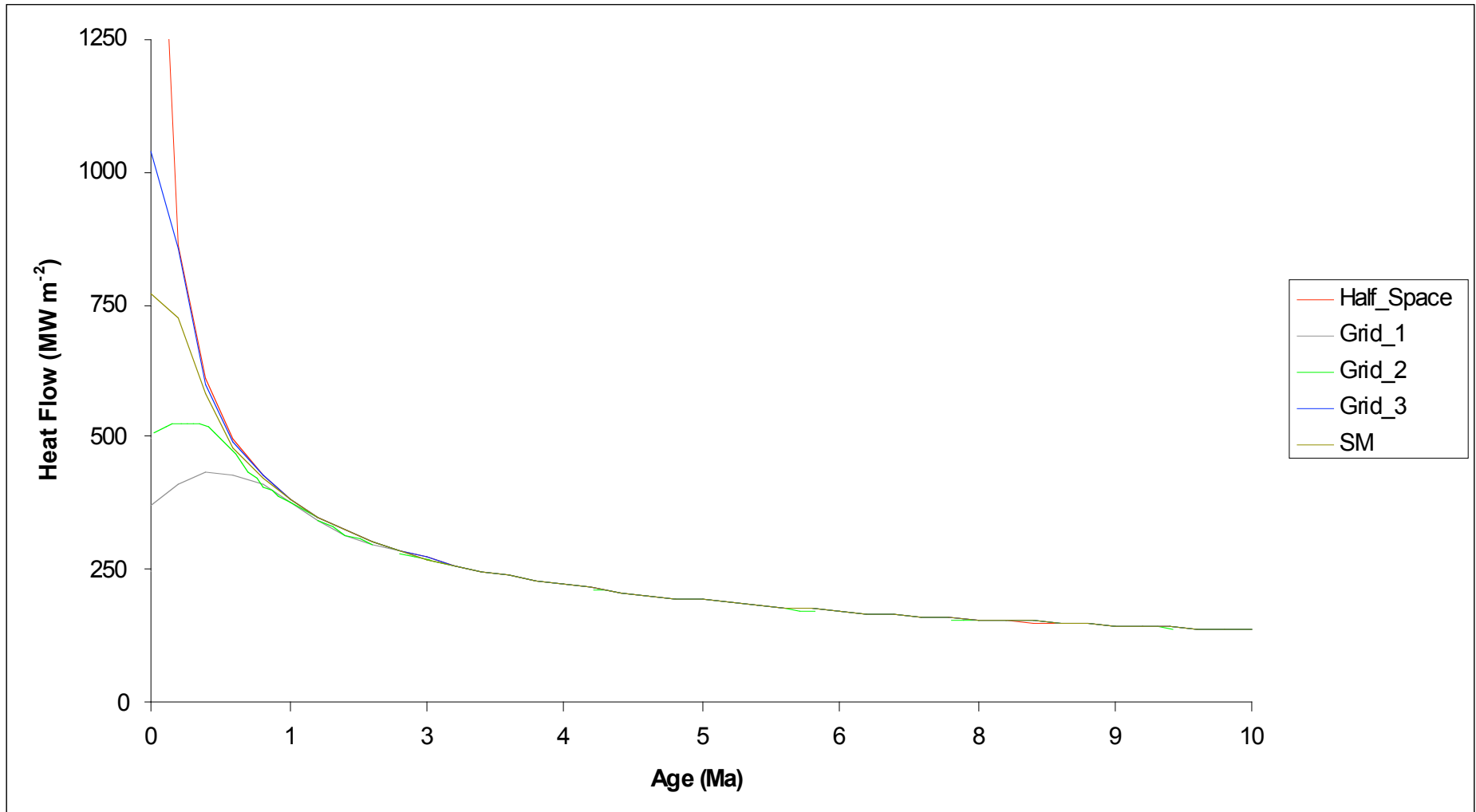


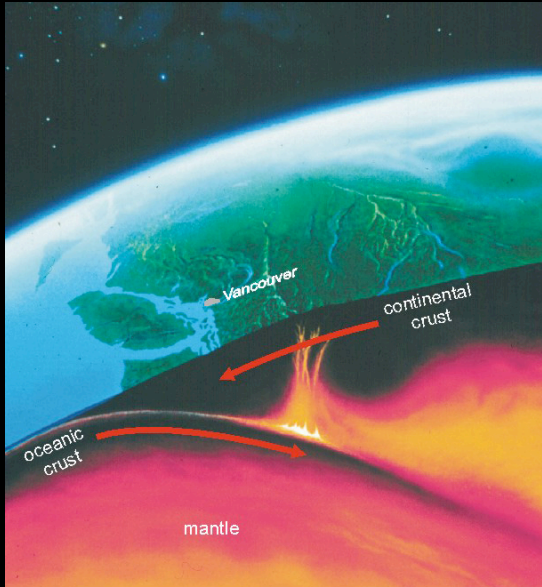
Mid Ocean Ridge-Simulations (2):



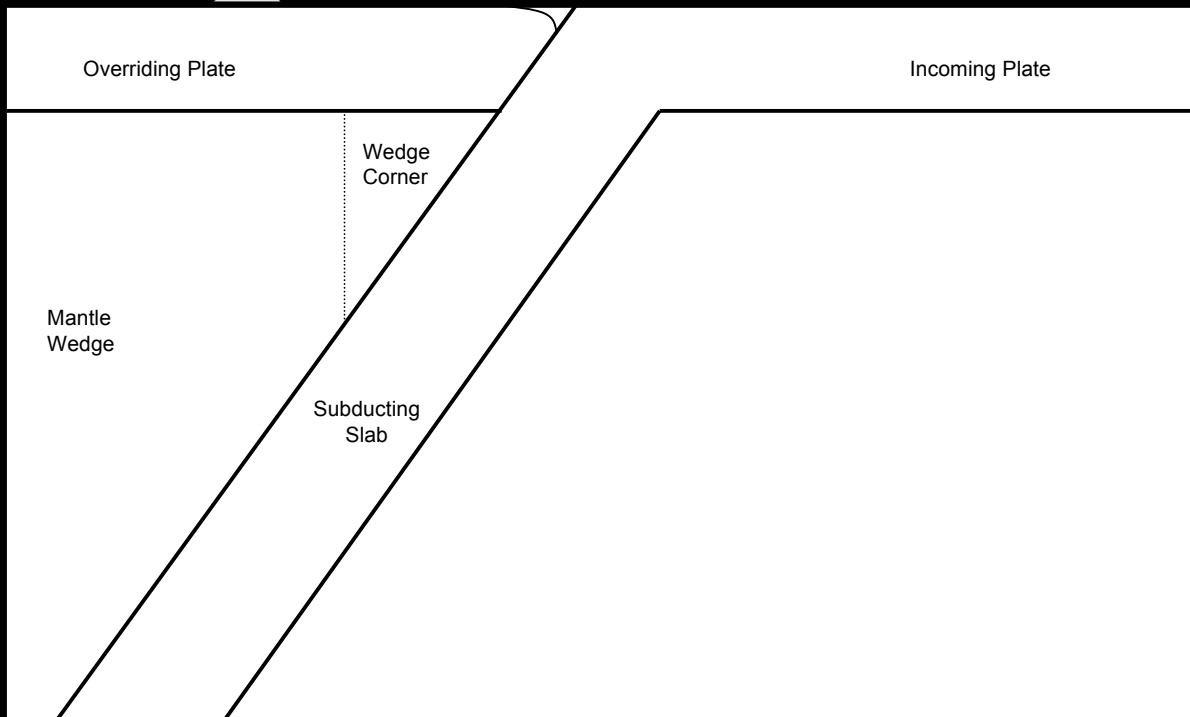


MOR – Surface Heat Flow:



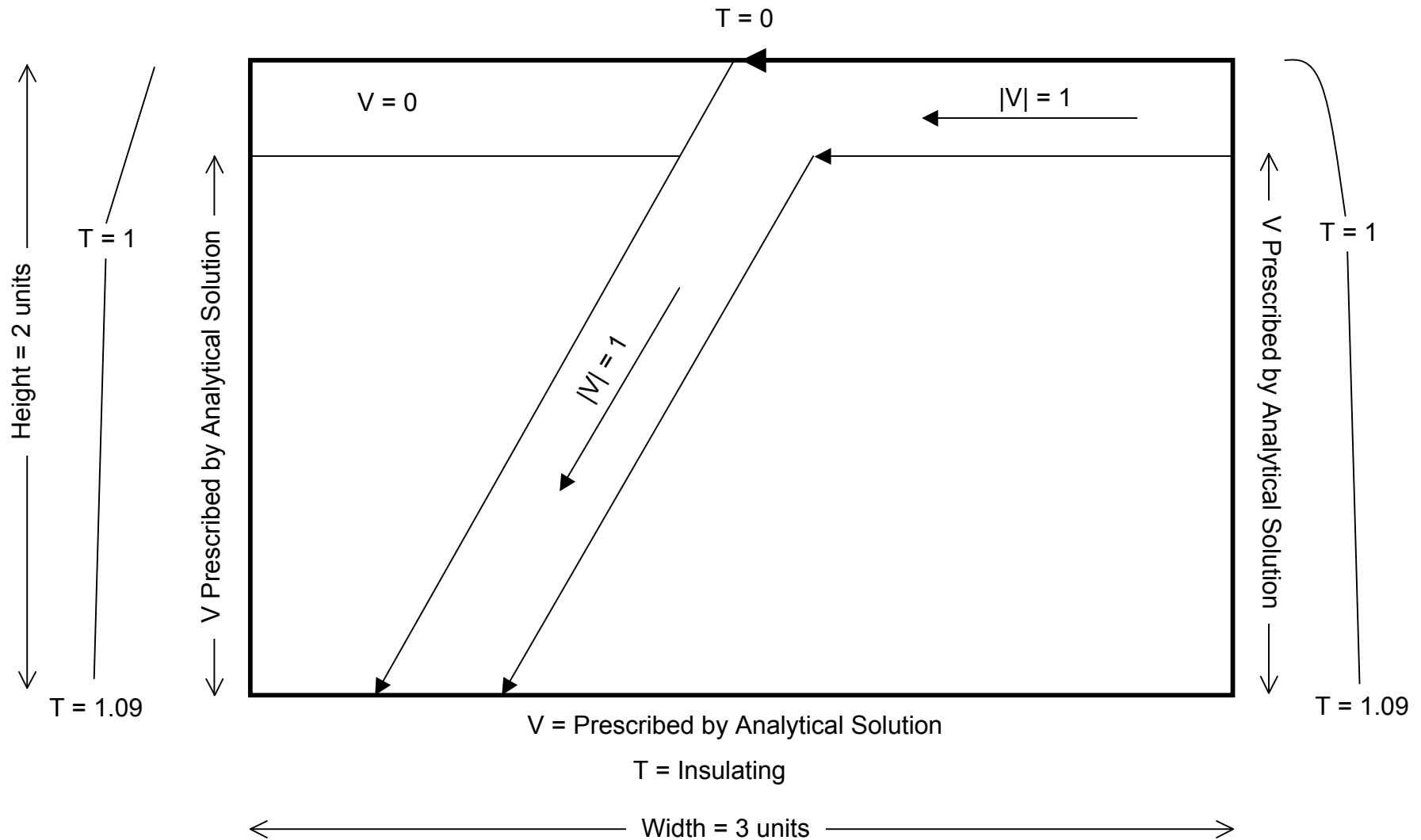


Subduction Zones:

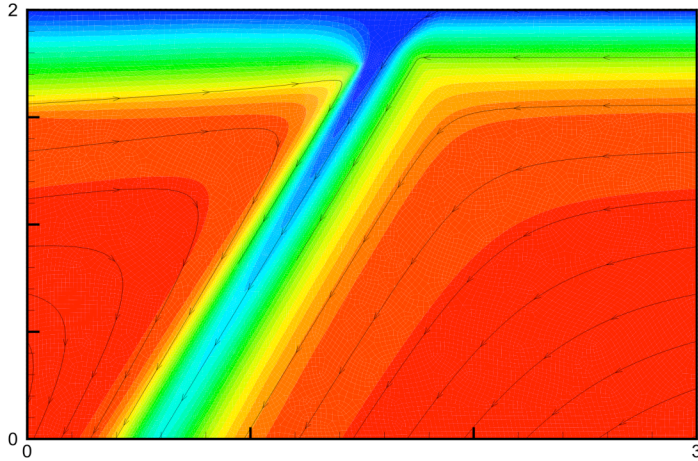




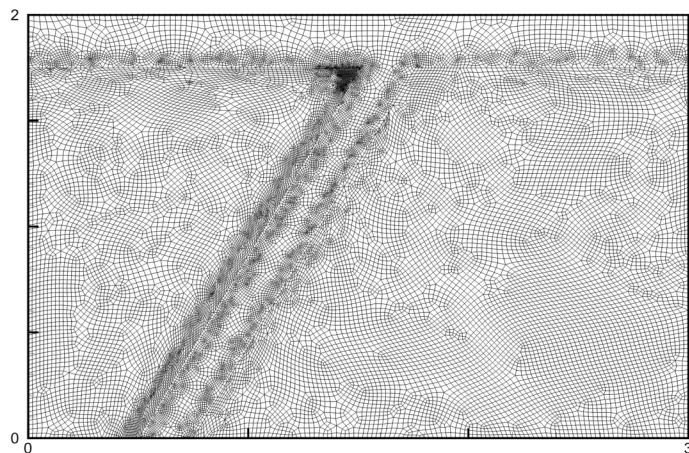
Boundary Conditions:



Results:



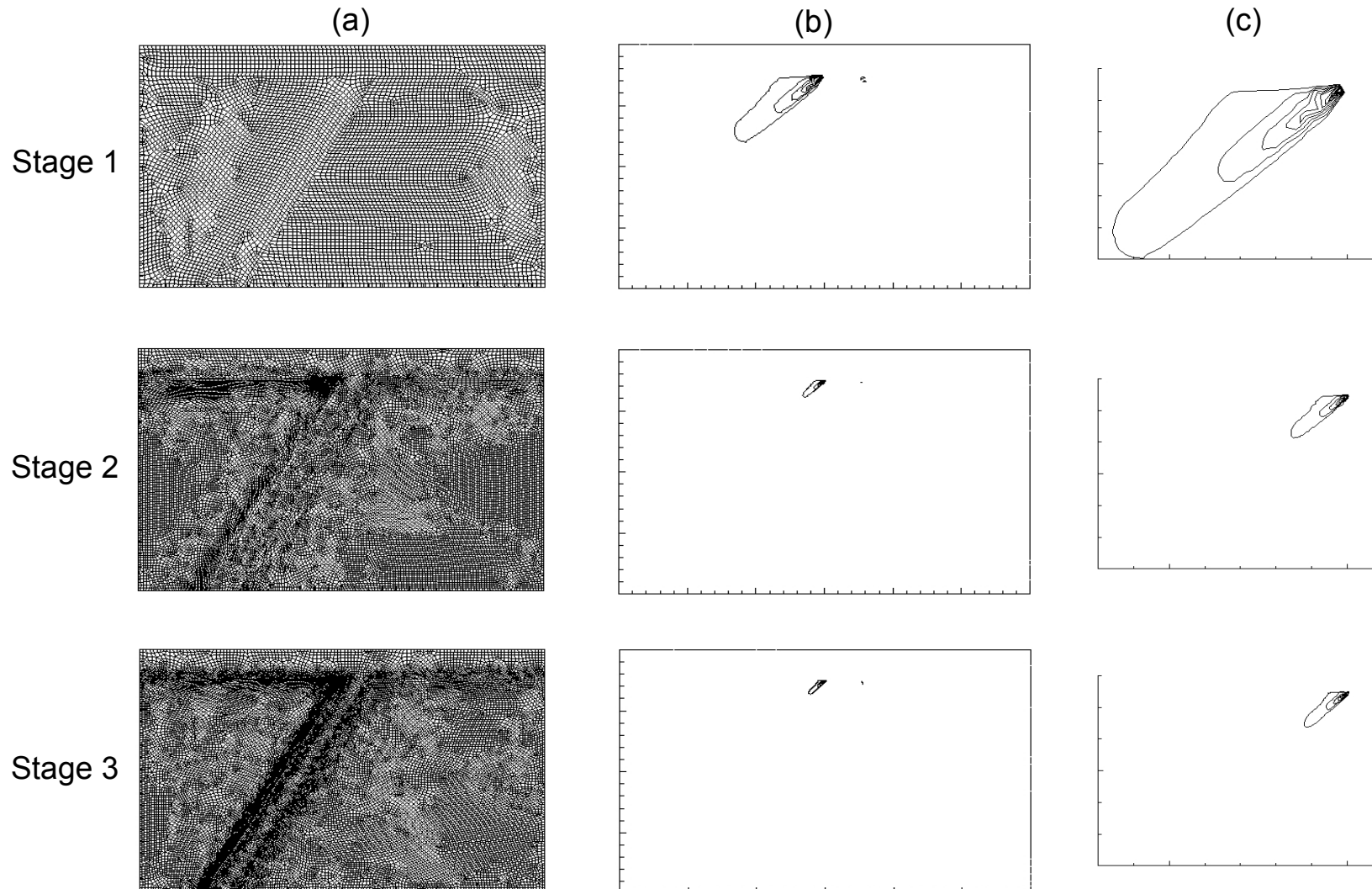
Thermal Field and
Velocity Stream-traces



Final Adapted Mesh

Solution Error:

Solution Error (comparison with analytical solution to a Newtonian corner flow problem)



2-D Conclusions.....



2-D Conclusions (general):

- An adaptive finite element procedure has been presented for solving convective heat transfer problems within the field of geodynamics.
- The method adapts the mesh **automatically** around regions of high solution gradient, yielding enhanced resolution of the associated flow features.
- Increases solution accuracy in a computationally efficient manner.



2-D Conclusions (thermal convection):

- The results obtained from purely thermal convective simulations are **extremely positive**.
- The error indicator presented has proven reliable and the adaptive procedure is shown to be robust.
- Predictions for heat transfer agree well with benchmark solutions, suggesting that the technique is valid and accurate.



2-D Conclusions (thermo-chemical convection):

- The AFEM provides a suitable means for increasing grid resolution in localized regions. This leads to a **reduction in numerical diffusion, and, hence, entrainment rates.**
- The grid based methods used for modelling the compositional field fail to achieve results that are consistent with other methods, even at the higher resolutions inherent to the AFEM.
- Our results suggest that an extension of this work to both tracer particle and marker chain methods would be a worthwhile exercise, with the higher spatial resolution yielded leading to the more accurate tracking of particles (or the marker chain), therefore generating results of greater accuracy.



2-D Conclusions (general):

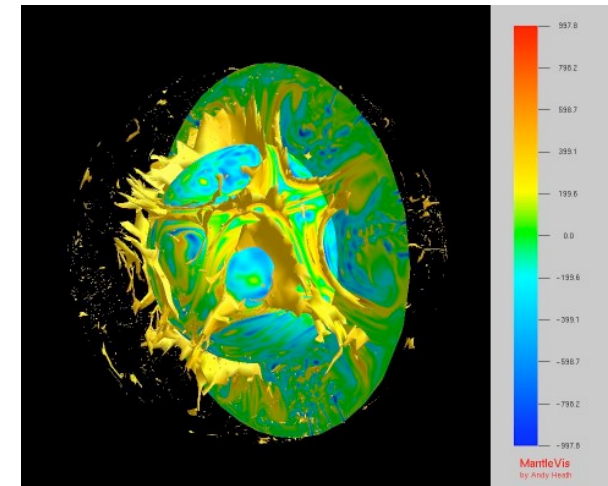
Perhaps the most important advantages of the AFEM are that:

1. The **unstructured** nature of the technique makes it easy to discretize the **complex geometries** encountered on Earth.
2. Nodes **automatically** cluster around zones of high solution gradient, without the need for *a priori* mesh generation.
3. The reduction in the number of degrees of freedom leads to a **decrease in computational memory demands and processing time** → complex problems can be solved efficiently, without excessive storage requirements.

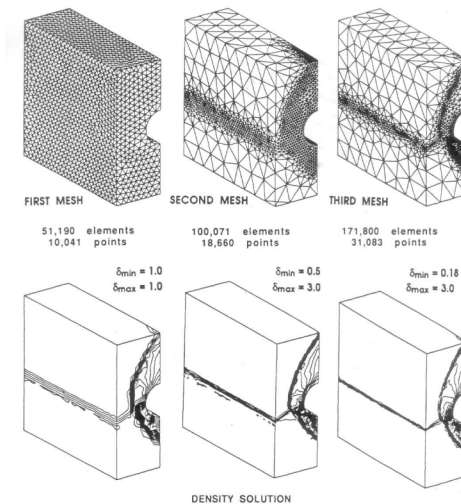
Grid Adaptivity In 3-D.....

Overview:

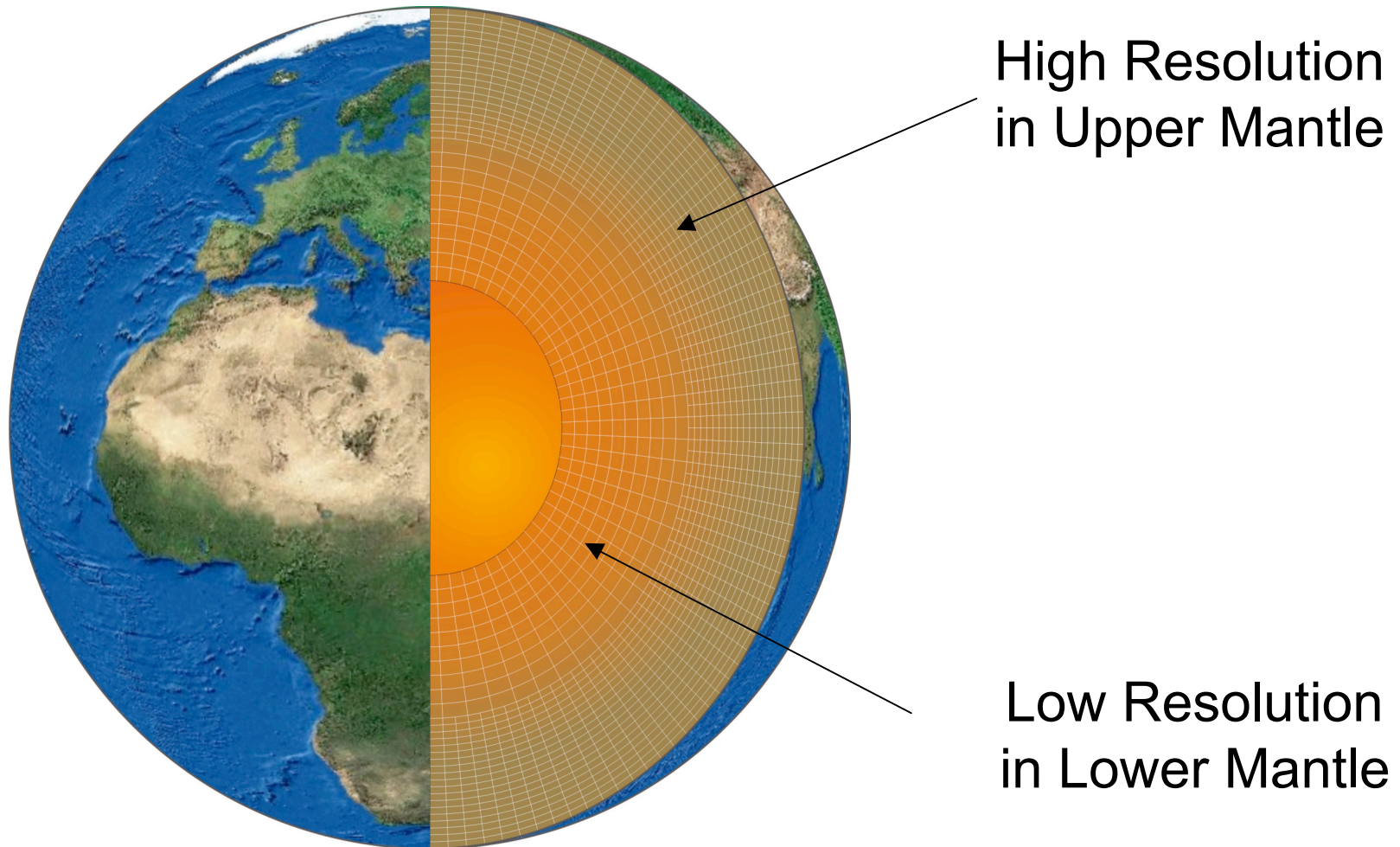
- Obvious benefits in 2-D. These benefits will be greater in 3-D.
- Don't know of any infinite Prandtl number, 3-D spherical, finite element codes that can deal with unstructured discretizations.
- Can the benefits be shown in 3-D within a suitable timescale?
- **Modify** an existing 3-D spherical-geometry code.
- Different techniques must be used - need a simplified method that is still beneficial in a computational and geodynamical sense.



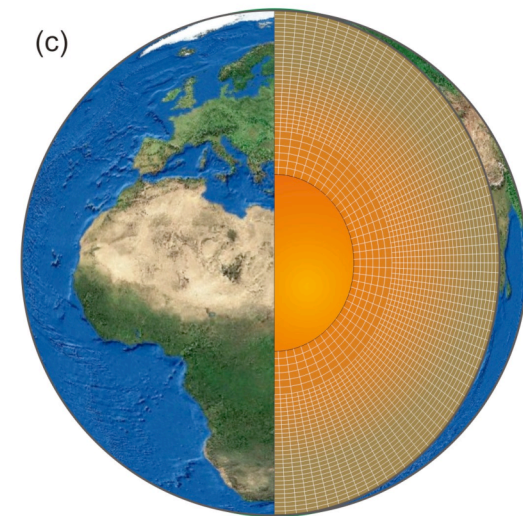
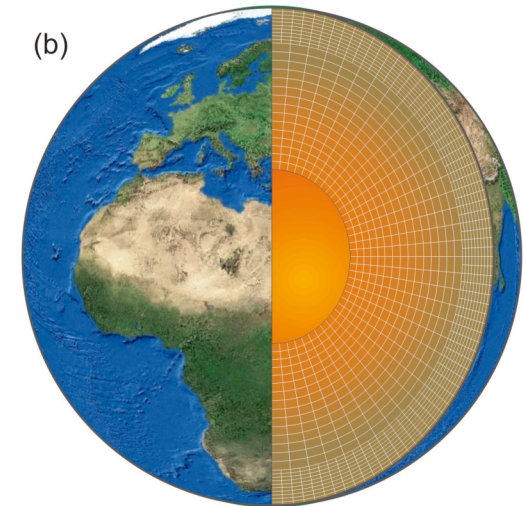
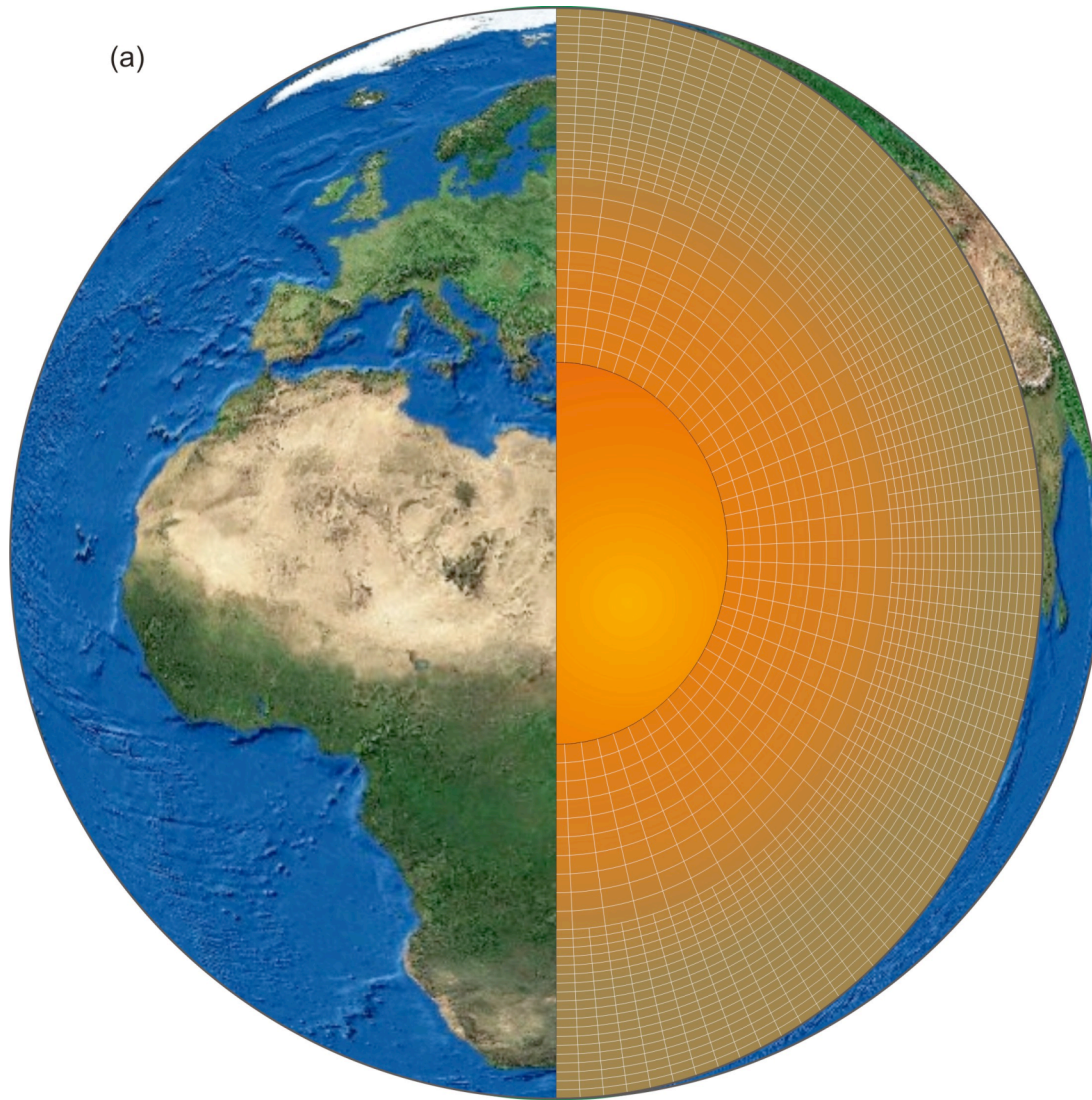
↑
Adaptivity?!



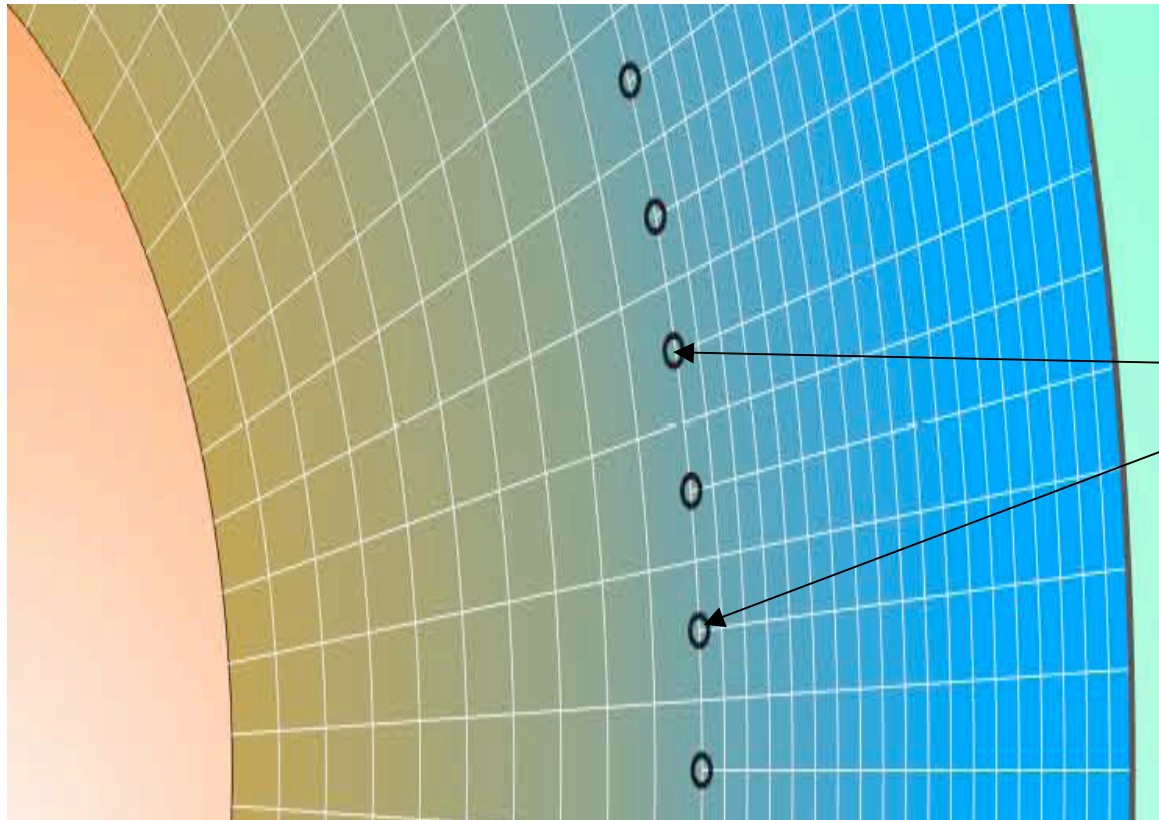
Radial Refinement / Variable Resolution:



Possible Discretizations:



The issue:

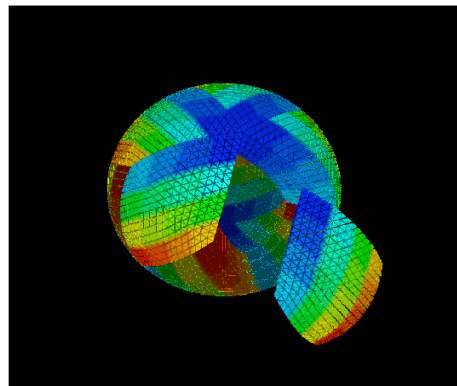


**Hanging
Nodes**



TERRA:

- TERRA is a well established **3D-spherical** finite element mantle dynamics code that was first developed by Baumgardner (1983).
- The code solves for momentum and energy balance in a spherical shell, with the inner radius being that of Earth's outer core and the outer radius corresponding to Earth's surface.



Example division of grid nodes to the clusters processors.



TERRA – The Governing Equations and Solution Strategies:

$$\nabla \cdot \tau - \nabla p + \rho g = 0$$

$$\nabla \cdot \mathbf{u} = 0$$

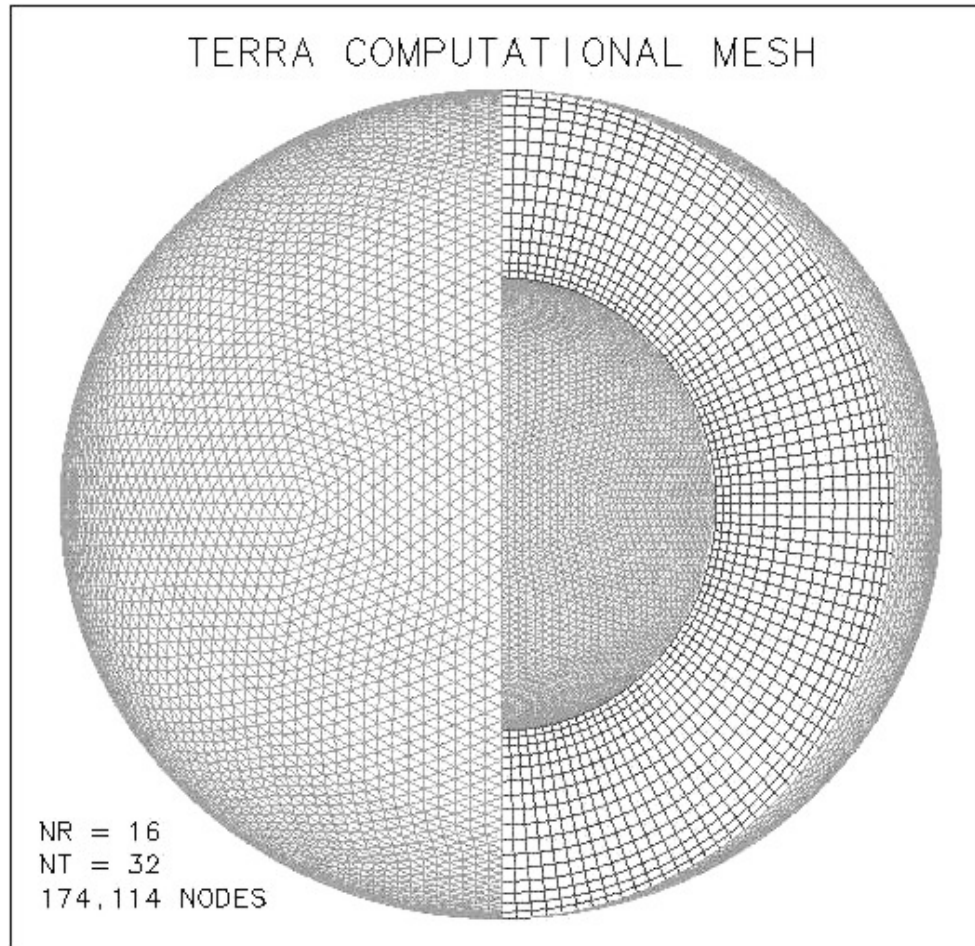
‘Uzawa’ pressure correction
conjugate gradient algorithm,
with multigrid inner solver.

$$\frac{\partial T}{\partial t} = -\nabla \cdot (T\mathbf{u}) - (\gamma - 1) T \nabla \cdot \mathbf{u} + \frac{[\tau : \nabla \mathbf{u} + \nabla \cdot (k\nabla T) + H]}{\rho c_V}$$

$$\tau = \mu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2I(\nabla \cdot \mathbf{u})}{3} \right]$$

A finite difference ‘flux form’ method.

TERRA – The Computational Grid:

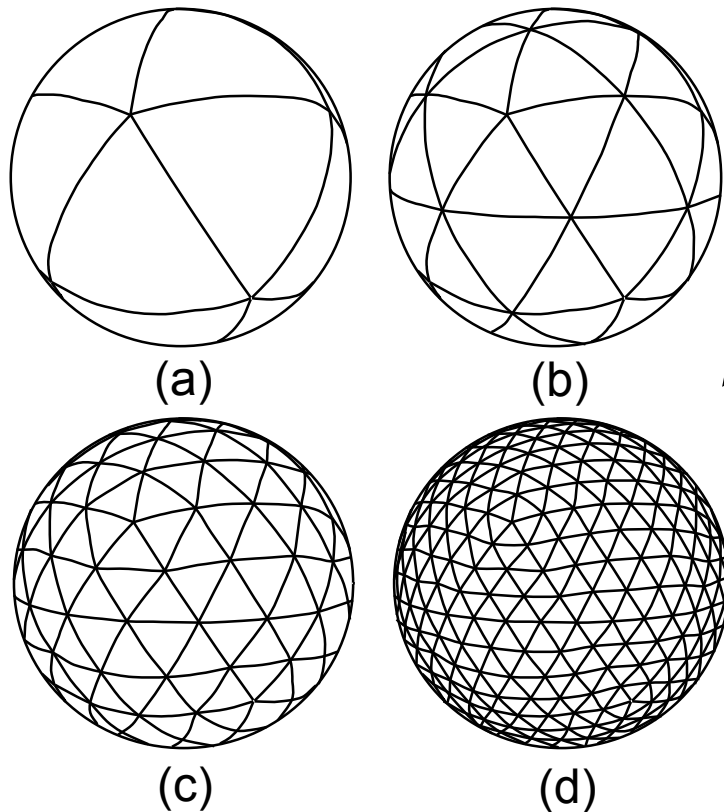


*Fully Structured
Quasi-Uniform Grid.*

Can code be modified to deal with uniform regions of variable resolution?

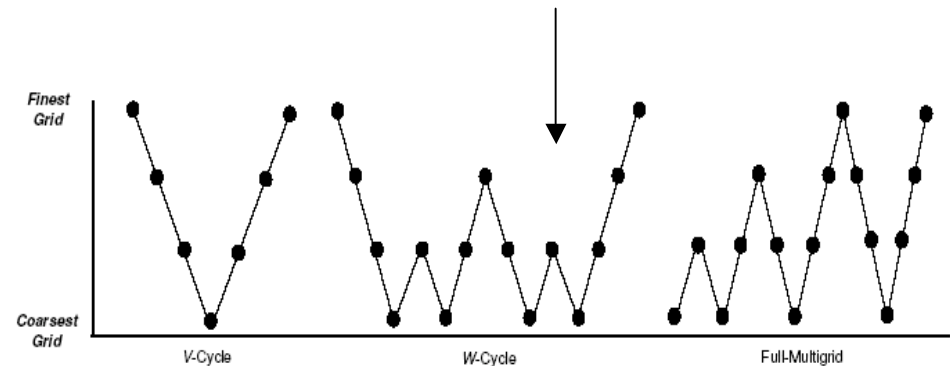
The Multigrid Concept:

A multigrid alternates the use of grids of various resolutions, achieving faster convergence than computations on fine grids and better accuracy than computations on coarse grids.

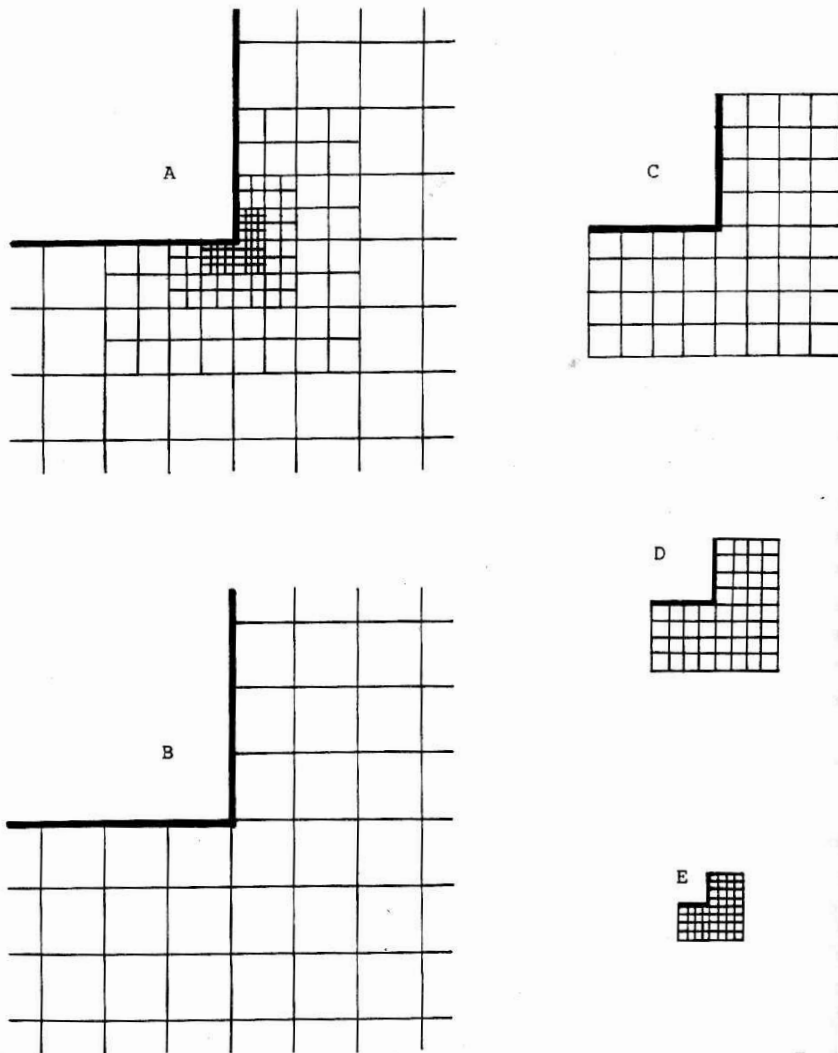


Various grid levels within TERRA's multigrid.

Multigrid Cycles



The 'Adaptive Multigrid':



Algorithm grows from the simple observation that the various grid levels used in the usual multigrid algorithms need not all extend over the same domain.

The effective mesh size in each neighbourhood will be that of the finest grid covering.

Structure provides a very efficient solution process by using its levels also as in a multigrid solver.

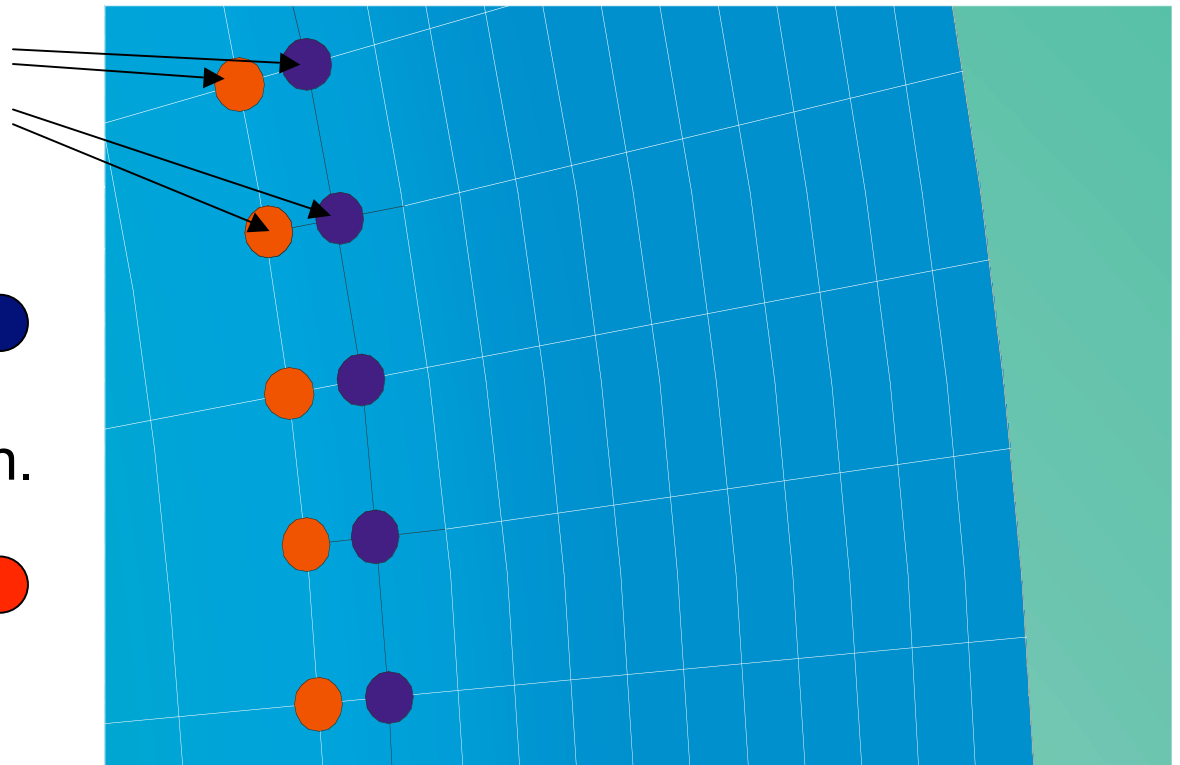
Essential prerequisite – *FAS scheme*.

Implementation (dealing with hanging nodes):

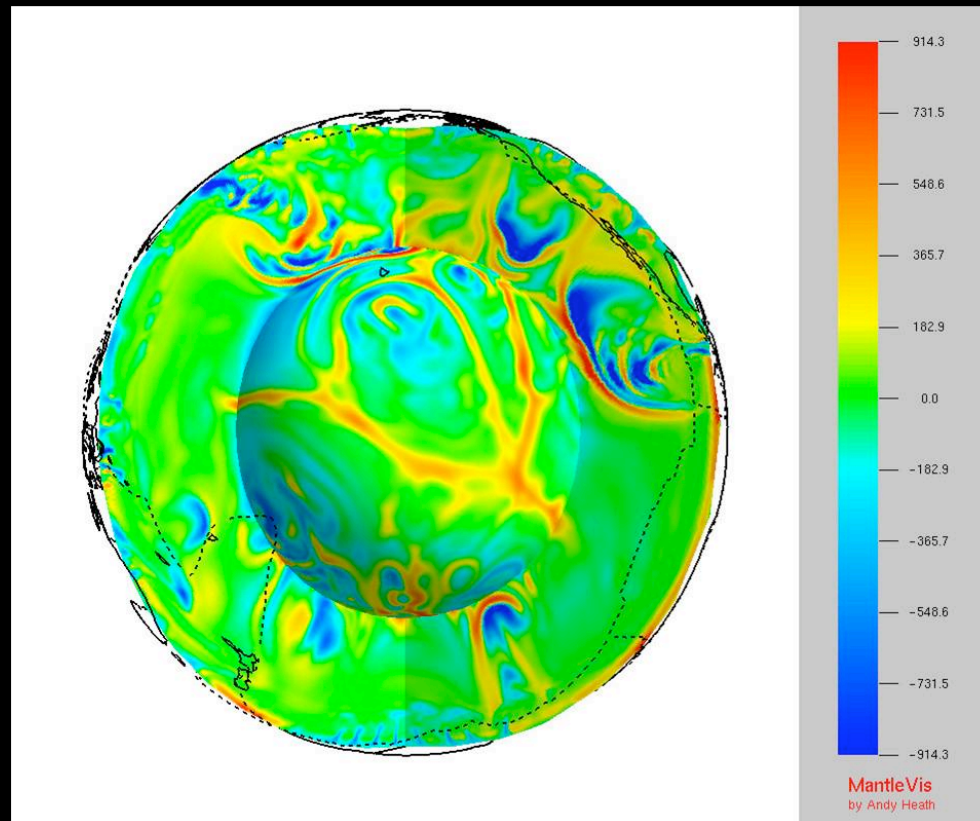
Series of 'Ghost nodes',
formed via interpolation.

Utilized as boundary
nodes during fine-grid
solution process - not
updated during relaxation.

Employed to ensure
solution continuity during
inter-grid transfers.



Validation.....

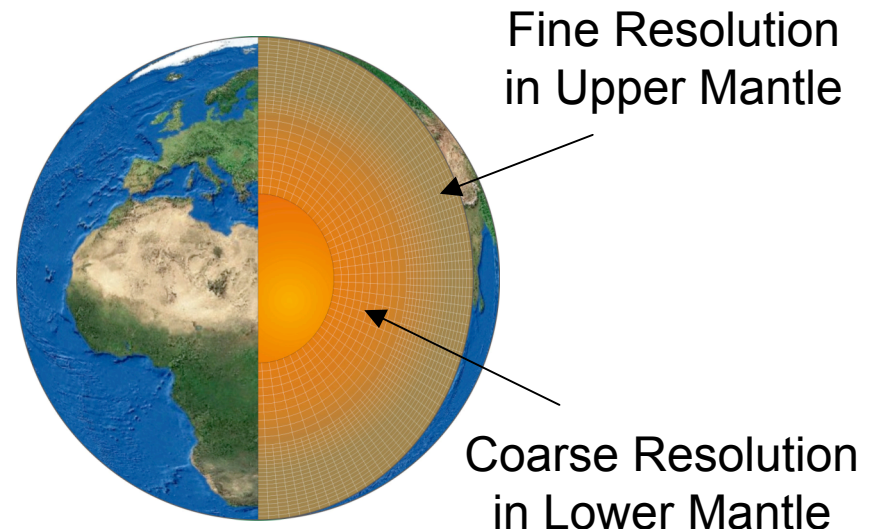


Test 1 – Finite Amplitude Convection:

Test 1: Finite amplitude thermal convection in isoviscous media.

Comparison with previously published results (e.g. Bercovici *et al.* 1989; Ratcliff *et al.* 1996; Zhong *et al.* 2000).

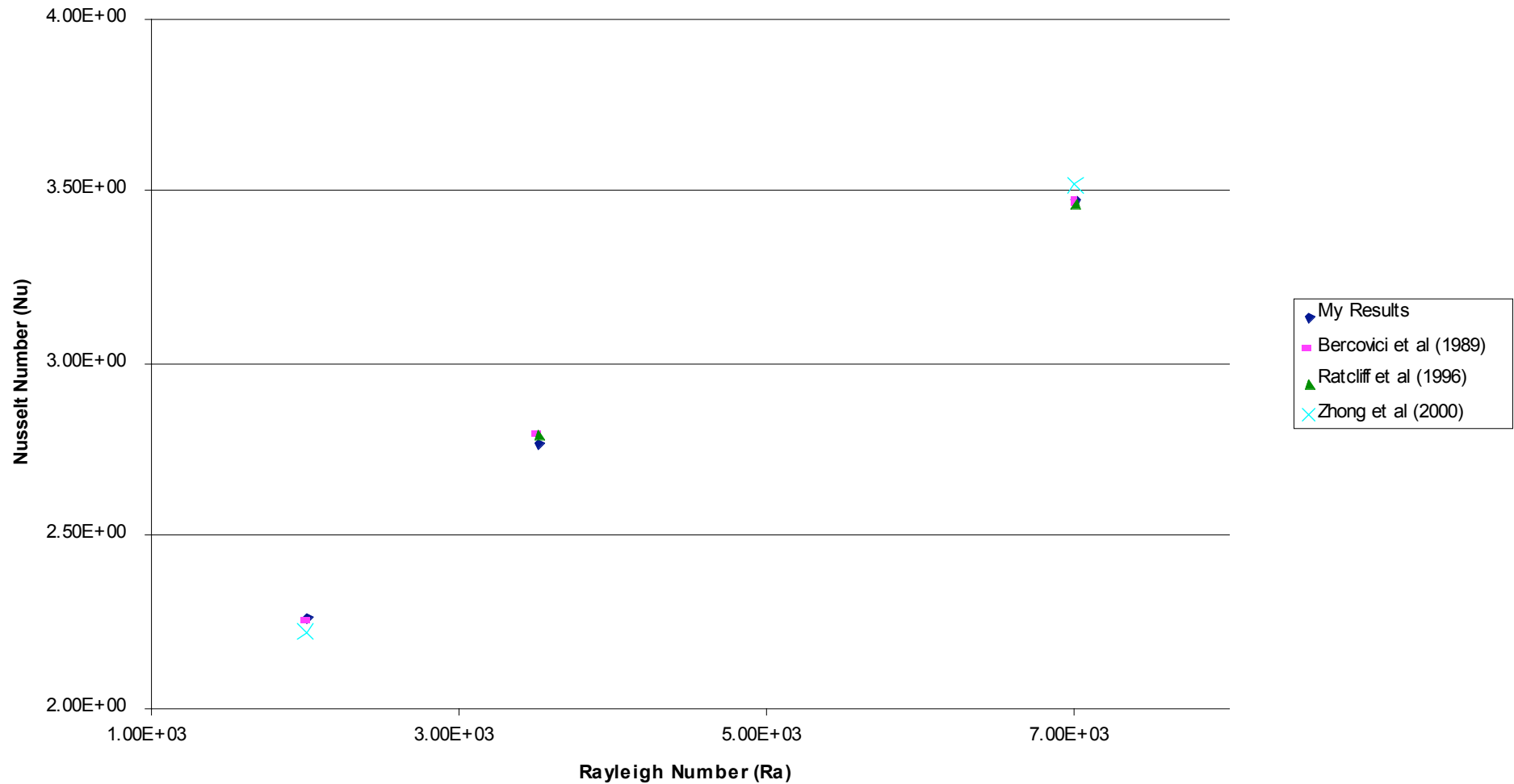
Verification of solvers for all three governing equations.





Test 1 – Finite Amplitude Tests:

Ra Vs Nu





Test 2 - Stokes Flow (Propagator Matrix):

Test 2: **Stokes Flow**, in which conservation equations of mass and momentum are solved for both isoviscous and layered viscosity structures.

Comparisons made with quasi-analytical solutions to these equations, derived via **propagator matrix** method (e.g. Hager & O'Connell, 1981).

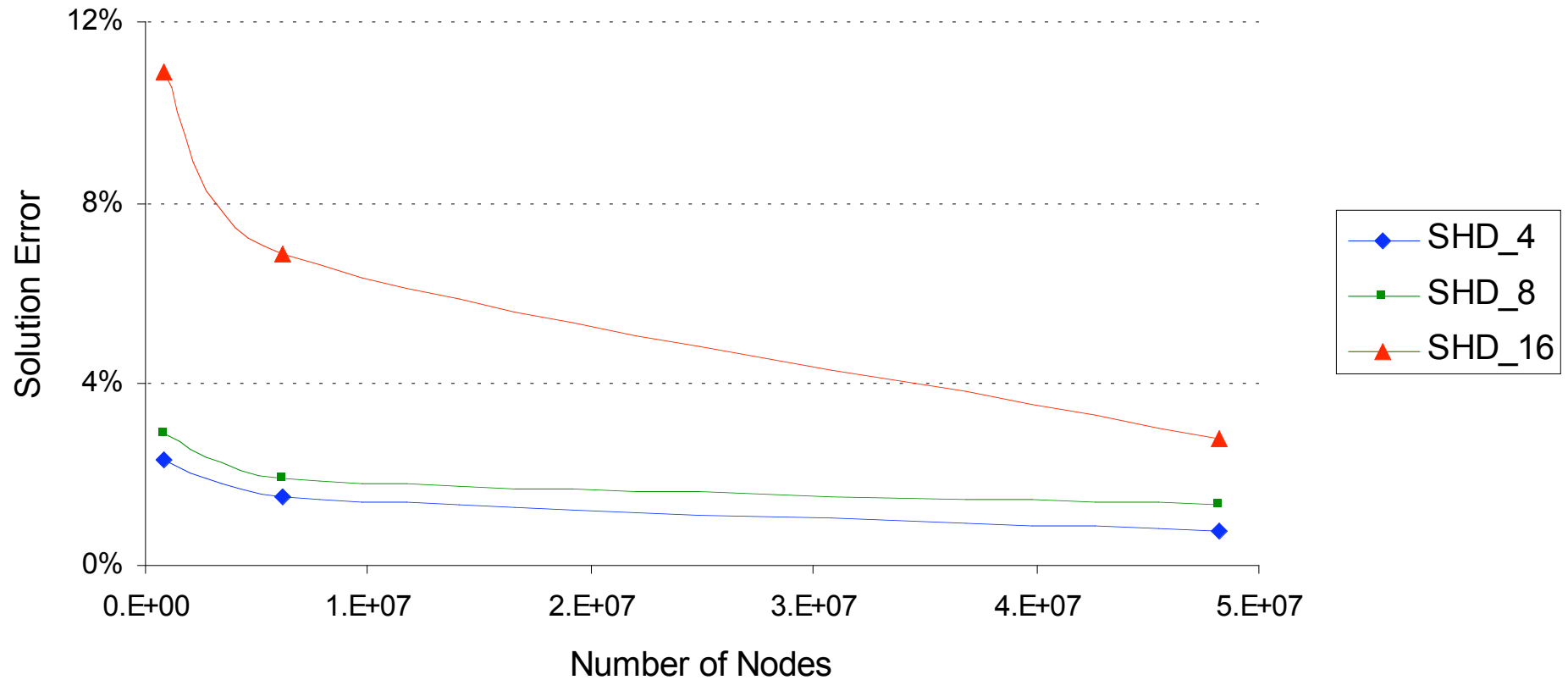
Specifically examine normalized **poloidal velocity coefficients**, in response to a spherical harmonic temperature perturbation in the mid-mantle.

<i>SHD</i>	<i>Outer Analytical</i>	<i>Outer Simulation</i>	<i>Inner Analytical</i>	<i>Inner Simulation</i>	<i>Outer Error</i>	<i>Inner Error</i>	<i>Total Error</i>
4	3.09E-10	3.12E-10	-3.39E-10	-3.37E-10	0.90%	0.61%	0.75%
8	6.84E-11	6.94E-11	-5.73E-11	-5.80E-11	1.44%	1.22%	1.33%
16	4.55E-12	4.67E-12	-1.90E-12	-1.84E-12	2.63%	2.91%	2.77%

<i>SHD</i>	<i>Outer Analytical</i>	<i>Outer Simulation</i>	<i>Inner Analytical</i>	<i>Inner Simulation</i>	<i>Outer Error</i>	<i>Inner Error</i>	<i>Total Error</i>
4	5.65E-10	5.69E-10	-4.40E-10	-4.37E-10	0.81%	0.75%	0.78%
8	1.27E-10	1.30E-10	-6.94E-11	-7.01E-11	2.35%	1.01%	1.68%
16	7.27E-12	7.70E-12	-1.97E-12	-2.04E-12	5.93%	3.50%	4.71%

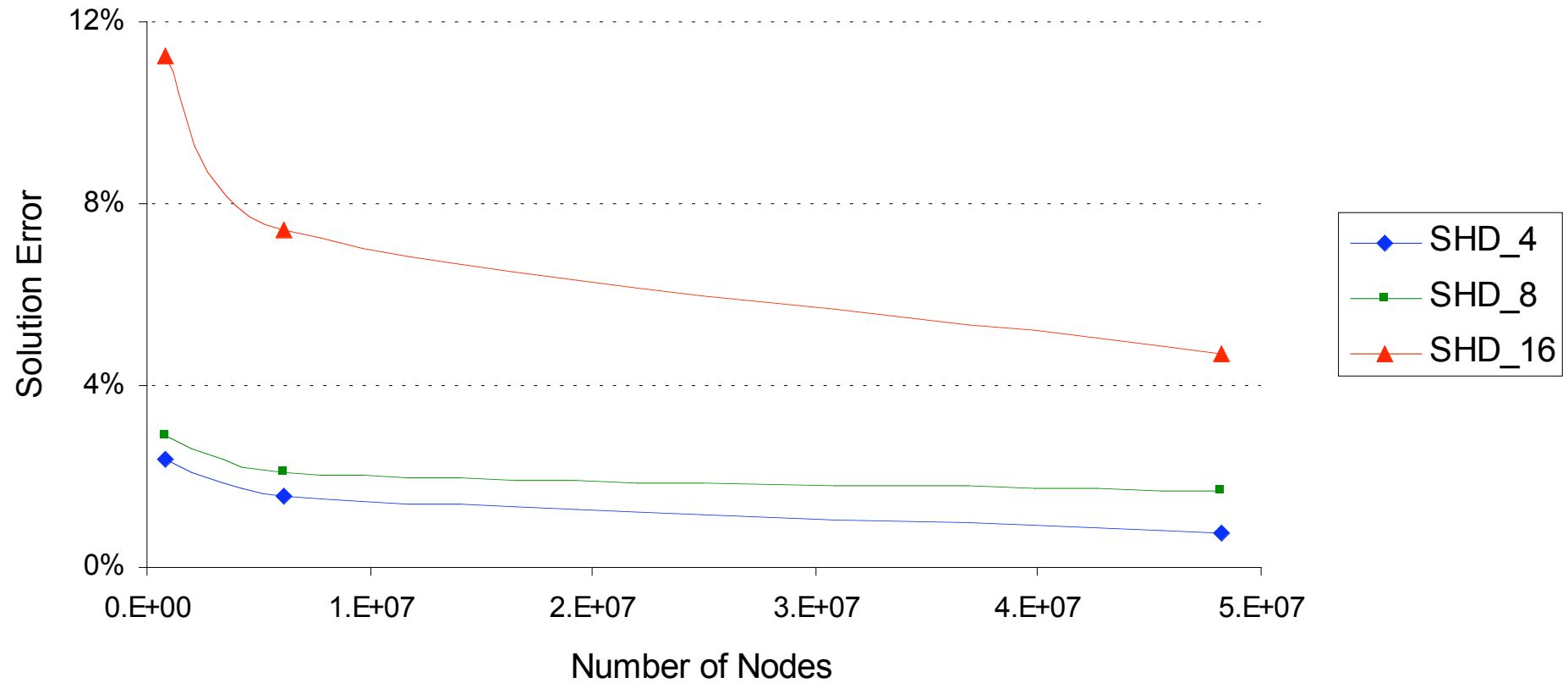


Uniform Viscosity - Convergence:





Layered Viscosity - Convergence:





3D Conclusions:

- Although highly simplified when compared to 2-D **adaptive remeshing**, the **adaptive multigrid** method investigated in 3-D appears to have similar advantages.
 - *Reduction in computational memory demands.*
 - *Reduction in processing time.*
- Validation tests demonstrate that methodology is **accurate**. Results agree well with previous work published in the literature.
- The method is capable of introducing **localized** high resolution zones to global 3-D spherical mantle convection models.



Publications:

1. Davies, D. R., Davies, J. H., Hassan, O., Morgan, K. & Nithiarasu, P. Adaptive Finite Element Methods in Geodynamics; Convection Dominated Mid-Ocean Ridge and Subduction Zone Simulations. *Int. J. Num. Meth. Heat Fluid Flow*. In Press.
2. Davies, D. R., Davies, J. H., Hassan, O., Morgan, K. & Nithiarasu, P. Investigations into the applicability of adaptive finite element methods for infinite Prandtl number thermal and thermo-chemical convection. *Geochem. Geophys. Geosyst.*, 8, Q05010, doi:10.1029/2006GC001470, 2007.