UNIVERSITY OF LEEDS

Anelastic dynamos and Giant Planet magnetic fields

Chris Jones
Department of Applied Mathematics

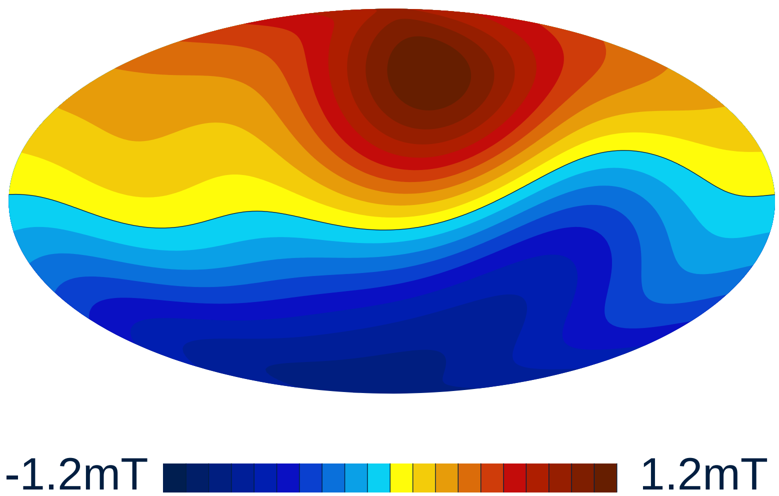
University of Leeds, UK

CIG HPC Dynamo Workshop, Boulder

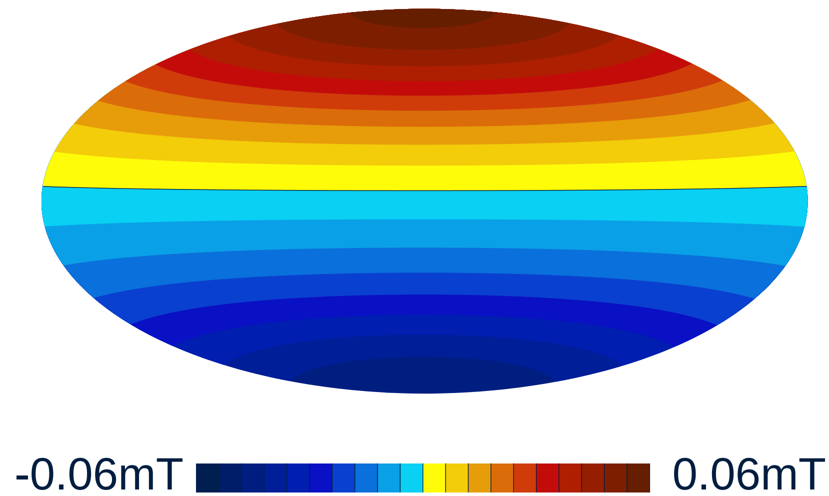
8th October 2012

Giant Planet Magnetic Fields

Radial field at Jupiter' surface



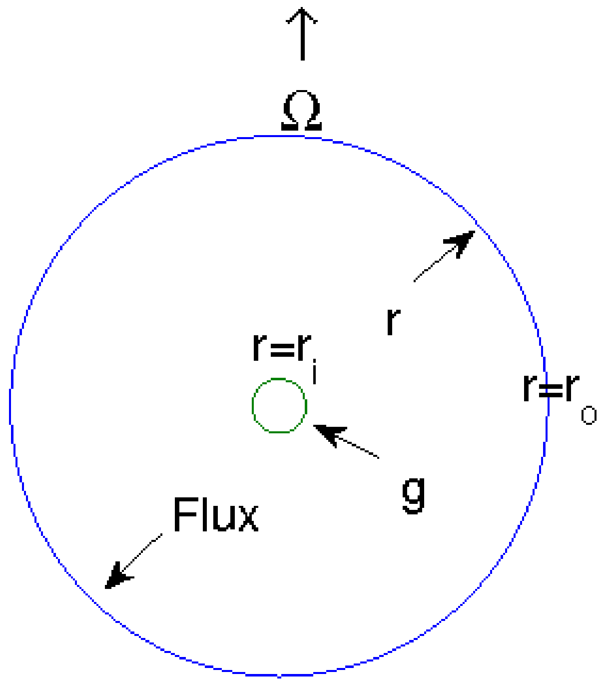
Saturn's radial field



Variable electrical conductivity due to metallic hydrogen. High pressure gives Earth-like conductivity out in $0 < r < 0.8R_{jup}$, $0 < r < 0.5R_{sat}$, falling to low values as surface approached.

Convection-driven but compressibility important. Substantial variation of density across the dynamo region.

Spherical geometry models



Compressible fluid shell lies between the inner core and outer boundary. Rotating about z-axis. Gravity radially inward. Length scale $d = r_o - r_i$ is gap-width from inner to outer boundary.

Anelastic approximation used, with the Lantz-Braginsky-Roberts formulation. Constant entropy boundaries or prescribed heat flux: Giant Planets driven by secular cooling, equivalent to a uniform entropy heat source.

Stable regions or convecting everywhere? Jupiter believed to be convecting everywhere, Saturn may have a stable zone in the interior due to condensation of helium.

Anelastic equation of motion: non-perfect gas

$$\rho = \bar{\rho} + \rho', \qquad P = \bar{P} + p', \qquad S = \bar{S} + s', \qquad T = \bar{T} + \vartheta.$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\bar{\rho}}\mathbf{j} \times \mathbf{B} - 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{F}_\nu - \frac{\nabla p'}{\bar{\rho}} - \nabla \Phi' - \frac{\rho' \nabla \bar{\mathbf{\Phi}}}{\bar{\rho}}.$$

Anelastic approximation: entropy drop across layer is small, $\Delta S / c_p = \epsilon \ll 1$.

Lantz/Braginsky/Roberts, define $\hat{p} = \frac{p'}{\bar{\rho}} + \Phi'$, and then

$$-\frac{\nabla p'}{\bar{\rho}} - \nabla \Phi' - \frac{\rho'}{\bar{\rho}} \nabla \bar{\Phi} = -\nabla \hat{p} - \hat{\mathbf{r}} \left[\frac{p'}{\bar{\rho}^2} \frac{d\bar{\rho}}{dr} + \frac{g \rho'}{\bar{\rho}} \right]$$

The density perturbation can be written

$$\rho' = \left(\frac{\partial \rho}{\partial S} \right)_p S' - \left(\frac{\partial \rho}{\partial S} \right)_p \left(\frac{\partial S}{\partial p} \right)_\rho p'$$

Since the basic state is close to adiabatic,

$$\frac{dS}{dr} = \left(\frac{\partial S}{\partial \rho}\right)_p \frac{d\bar{\rho}}{dr} + \left(\frac{\partial S}{\partial p}\right)_\rho \frac{d\bar{p}}{dr} \approx 0$$

which means the pressure perturbation term is negligible, so

$$-\frac{\nabla p'}{\bar{\rho}} - \nabla\Phi' - \frac{\rho'}{\bar{\rho}}\nabla\bar{\Phi} = -\nabla\hat{p} - \hat{\mathbf{r}}\left(\frac{\partial \rho}{\partial S}\right)_p \frac{gS'}{\bar{\rho}}$$

Finally, using Maxwell's thermodynamic relations and the hydrostatic equilibrium of the reference state, we get

$$\left(\frac{\partial \rho}{\partial S}\right)_p = -\rho^2\left(\frac{\partial T}{\partial p}\right)_S = \frac{\rho}{g}\frac{d\bar{T}}{dr}, \quad \text{so the equation of motion is}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\hat{p} - 2\boldsymbol{\Omega}\times\mathbf{u} + \frac{1}{\bar{\rho}\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} + \mathbf{F}_\nu - \hat{\mathbf{r}}\frac{d\bar{T}}{dr}S'$$

Entropy and Continuity

Assume turbulent entropy diffusion dominates radiation and conduction.

$$\bar{\rho}\bar{T}\frac{\mathcal{D}S}{\mathcal{D}t} = \nabla \cdot \kappa\bar{\rho}\bar{T}\nabla S + Q_\nu + Q_j$$

where

$$Q_\nu = \sigma_{ij}\frac{\partial u_i}{\partial x_j}, \qquad \sigma_{ij} = \nu\bar{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla \cdot \mathbf{u}\right),$$

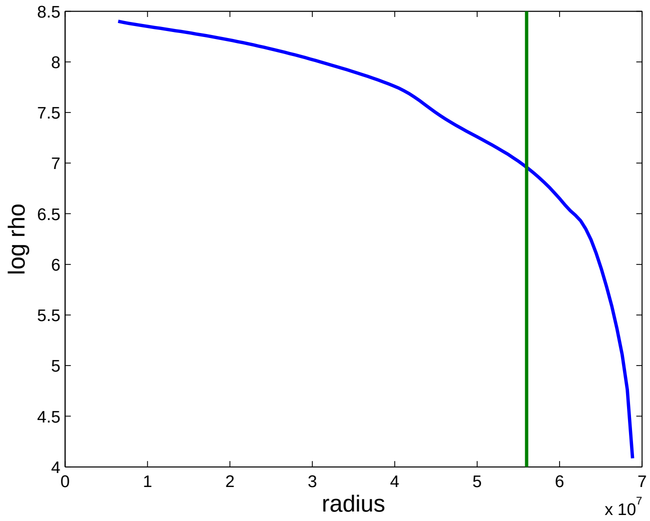
$$Q_j = \eta\mu_0\mathbf{j}^2, \qquad \mu_0\mathbf{j} = \nabla \times \mathbf{B}.$$

Fluid speed is $O(\epsilon^{1/2})$ compared to sound and Alfvén speeds, and

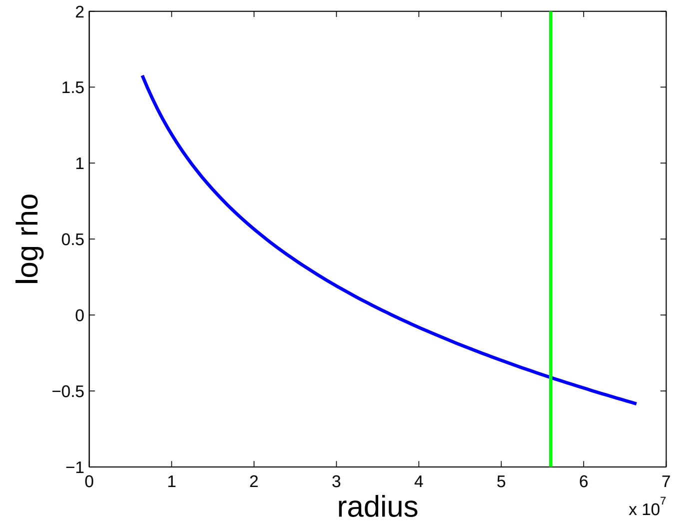
$$\nabla \cdot \bar{\rho}\mathbf{u} = 0, \qquad \mathbf{u} = \frac{1}{\bar{\rho}}\nabla \times \nabla \times \mathbf{r}\mathcal{P}\bar{\rho} + \frac{1}{\bar{\rho}}\nabla \times \mathbf{r}\mathcal{T}\bar{\rho},$$

is the toroidal-poloidal expansion.

Reference state



French et al model



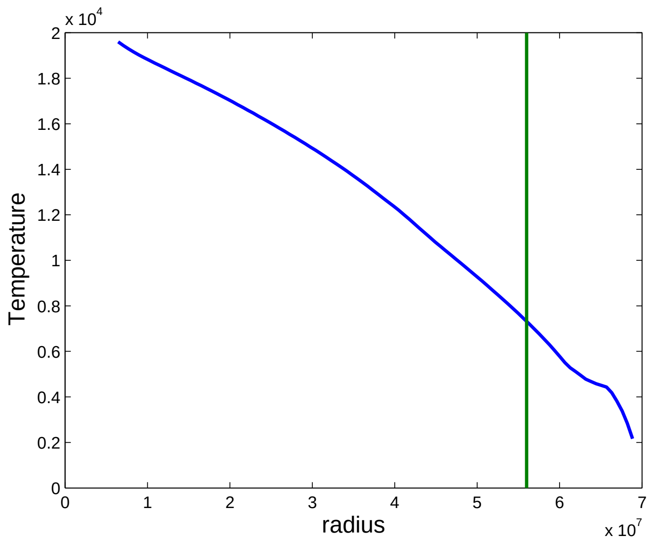
$n = 2$ polytrope model.

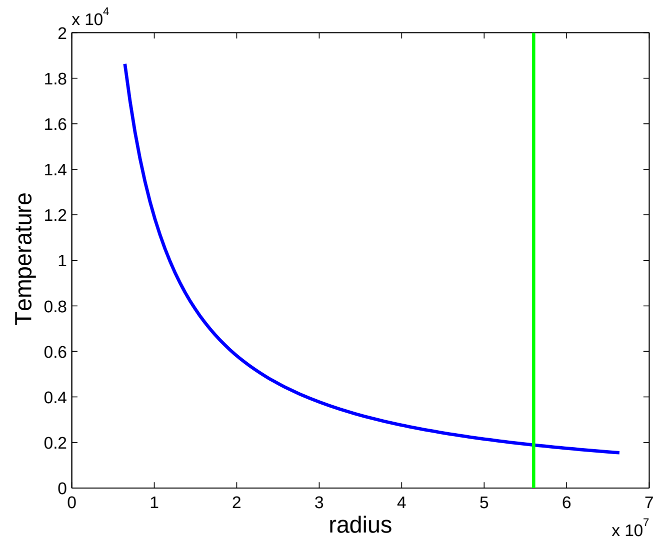
Assume reference state is isentropic. French et al 2012 model gives $\log p(\log \rho)$ and $\log T(\log \rho)$ as splines, then solve hydrostatic equation.

Inner core at 6,170 km from centre, core mass 1.1% of Jupiter's mass.

Have to cut off the model to avoid very low densities. Severe truncation case cuts off 3000km below the 1 bar level, (density ratio 20) moderate truncation cuts off 700 km below (density ratio 150).

Basic state temperature structure





The temperature gradient is much steeper near the surface for the Jupiter model compared to the polytrope. This amplifies the driving near the outer boundary, adding to the computational difficulty.

French et al. also give the electrical conductivity, but before this was published, Gomez-Perez et al. 2010 was used.

Even for the severe truncation, the electrical conductivity is essentially zero in the ignored region.

Dimensionless Parameters

Input Parameters based on quantities at $(r_o + r_i)/2$:

Ekman number $E = \nu/\Omega d^2$

Rayleigh number $Ra = \dfrac{T_c d^2 \Delta S}{\nu\kappa}$

Prandtl number $Pr = \nu/\kappa$

Magnetic Prandtl number $Pm = \nu/\eta$

η magnetic diffusivity, ν kinematic viscosity, κ is the thermal diffusivity, $d = r_o - r_i$.

Input reference state density, temperature, diffusivity profiles. Mostly used constant κ and ν, but other choices possible.

Parameters output from model: Rossby number $Ro = U/\Omega d$, Magnetic Reynolds number, $Rm = Ud/\eta$, etc.

Viscous terms

For constant kinematic viscosity the dimensionless form of

$$\mathbf{F}_\nu = \left[\frac{1}{\bar\rho}\frac{\partial}{\partial x_j}\bar\rho\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3\bar\rho}\frac{\partial}{\partial x_i}\left(\bar\rho\frac{\partial u_j}{\partial x_j}\right)\right] = \hat{\mathbf{F}}_\nu - \frac{1}{\bar\rho}\nabla\times\nabla\times\bar\rho\mathbf{u}.$$

The dimensionless equation of motion is then written

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\bar\rho}\nabla\times\nabla\times\bar\rho\mathbf{u} = -\nabla\hat p + \mathbf{N}_v$$

where

$$\mathbf{N}_v = \mathbf{u}\times\boldsymbol{\omega} + Pm\left[\hat{\mathbf{F}}_\nu - \frac{2}{E}\hat{\mathbf{z}}\times\mathbf{u} - \frac{Pm}{Pr}RaS\frac{d\bar T}{dr}\hat{\mathbf{r}} + \frac{1}{E\,\rho}(\nabla\times\mathbf{B})\times\mathbf{B}\right],$$

$$\hat{\mathbf{F}}_\nu = \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}) + \frac{1}{\bar\rho}(\nabla\bar\rho\times\boldsymbol{\omega}) - \frac{\mathbf{u}}{\bar\rho}\nabla^2\bar\rho + \frac{2}{3}\hat{\mathbf{r}}u_r\xi^2, \quad \xi = \frac{1}{\bar\rho}\frac{d\bar\rho}{dr}.$$

is included with the nonlinear terms. Note r-derivatives of reference state density required here.

Viscous heating terms

Complicated terms: how to prevent them slowing down the code?

Split the viscous heating term into two parts

$$Q_v = Q_v^{(1)} + Q_v^{(2)},$$

$$Q_v^{(1)} = 2\left(\frac{\partial u_r}{\partial r}\right)^2 - \frac{2}{3}(\xi u_r)^2 + 2\left(\frac{\partial u_r}{\partial r} + u_r(\xi + \frac{1}{r}) + q_v^{(1)}\right)^2$$
$$+2\left(q_v^{(1)} + \frac{u_r}{r}\right)^2,$$

$$Q_v^{(2)} = \left(2\frac{\partial u_\theta}{\partial r} - \omega_\phi\right)^2 + \left(2\frac{\partial u_\phi}{\partial r} + \omega_\theta\right)^2 + \left(2q_v^{(2)} + \omega_r\right)^2$$

$$q_v^{(1)} = \frac{u_\theta \cos\theta}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}, \qquad q_v^{(2)} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi \cos\theta}{r\sin\theta},$$

Already transforming $\mathbf{u}$ and $\boldsymbol{\omega}$ to physical space, so only need additional transforms for radial derivative of $\mathbf{u}$ and the two q_v's.

Equations for $\mathcal{P}$, $\mathcal{T}$, $\mathcal{P}_\mathcal{B}$ and $\mathcal{T}_\mathcal{B}$

$$\hat{\mathbf{r}} \cdot \nabla \times \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \times \nabla \times \bar{\rho} \mathbf{u} \right) = \hat{\mathbf{r}} \cdot \nabla \times \mathbf{N}_v$$

with

$$\mathbf{u} = \frac{1}{\bar{\rho}} \nabla \times \nabla \times \mathbf{r} \mathcal{P} \bar{\rho} + \frac{1}{\bar{\rho}} \nabla \times \mathbf{r} \mathcal{T} \bar{\rho},$$

gets rid of the poloidal part $\mathcal{P}$, leaving

$$\frac{-L^2}{r} \left(\frac{\partial \mathcal{T}}{\partial t} - \frac{1}{\rho} \nabla^2 (\rho \mathcal{T}) \right) = \hat{\mathbf{r}} \cdot \nabla \times \mathbf{N}_v, \; L^2 = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2}.$$

Since we expand in spherical harmonics, this simplifies to a second order parabolic equation,

$$\frac{\partial \mathcal{T}}{\partial t} - \frac{1}{\rho} \nabla^2 (\rho \mathcal{T}) = \frac{r}{\ell(\ell+1)} \hat{\mathbf{r}} \cdot \nabla \times \mathbf{N}_v$$

with one boundary condition on $\mathcal{T}$ at each boundary.

Similarly, for the magnetic field

$$\mathbf{B} = \nabla \times \nabla \times \mathbf{r}\mathcal{P}_{\mathcal{B}} + \nabla \times \mathbf{r}\mathcal{T}_{\mathcal{B}},$$

and we separate out the toroidal part of the induction equation with

$$\hat{\mathbf{r}} \cdot \nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \eta(r) \nabla \times \mathbf{B} \right) = \hat{\mathbf{r}} \cdot \nabla \times \nabla \times (\mathbf{u} \times \mathbf{B}) = \hat{\mathbf{r}} \cdot \nabla \times \mathbf{N}_B$$

to get

$$\frac{\partial \mathcal{T}_{\mathcal{B}}}{\partial t} - \nabla \cdot (\eta \nabla \mathcal{T}_{\mathcal{B}}) = \frac{r}{\ell(\ell+1)} \hat{\mathbf{r}} \cdot \nabla \times \mathbf{N}_b.$$

The poloidal parts are a little more complicated. For the velocity we take the r-component of the double curl of the equation of motion,

$$\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \times \nabla \times \bar{\rho} \mathbf{u} \right) = \hat{\mathbf{r}} \cdot \nabla \times \nabla \times \mathbf{N}_v$$

The toroidal parts are all zero, but at first sight the viscous term, involving curl^6, looks unpleasant.

Remarkably, even with $\bar{\rho}$ an essentially arbitrary function of r, it reduces to a fourth order operator in r which factorises into two quadratic factors,

$$\frac{\partial \mathcal{P}}{\partial t} - \frac{1}{\rho}\nabla^2(\rho\mathcal{P}) = G,$$

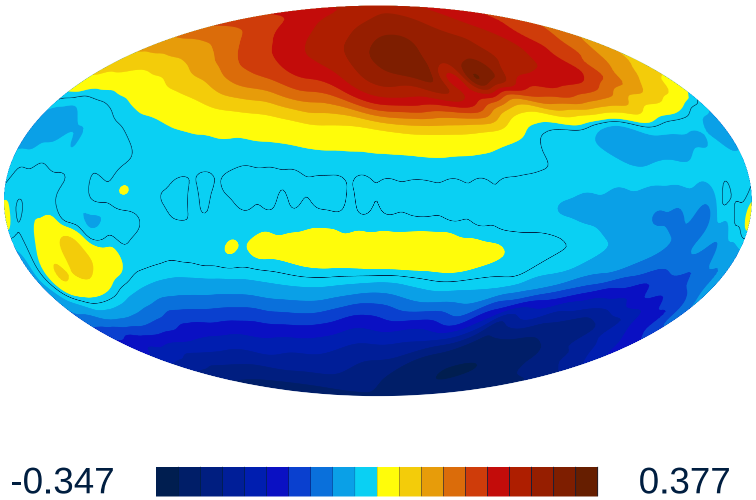
$$\nabla^2 G + \frac{1}{r}\frac{\partial}{\partial r}(r\xi G) = -\frac{r}{\ell(\ell+1)}\hat{\mathbf{r}}\cdot\nabla\times\nabla\times\mathbf{N}_v.$$

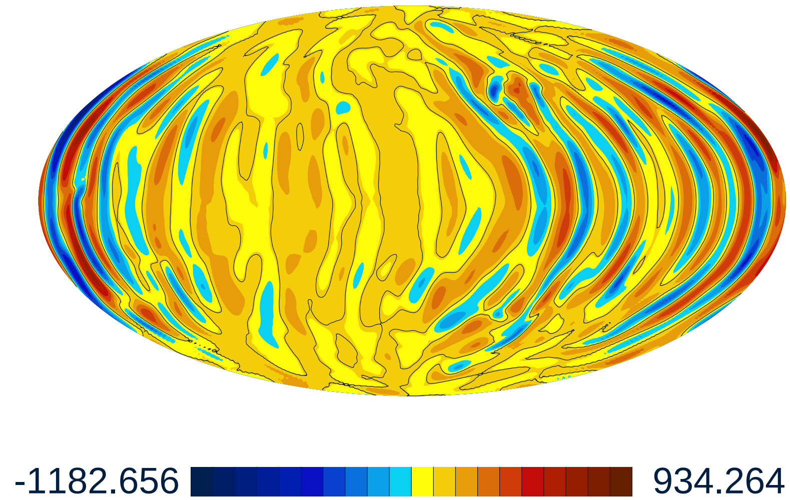
To satisfy the four boundary conditions on $\mathcal{P}$, use an influence matrix method, that is precompute two solutions satisfying two point boundary conditions on G with zero right hand side, and add on a suitable combination of these to ensure all four bc's on $\mathcal{P}$ are satisfied.

Poloidal component of induction is solved the same way.

Entropy equation has the same form as the toroidal equations.

Jupiter Model Dynamo





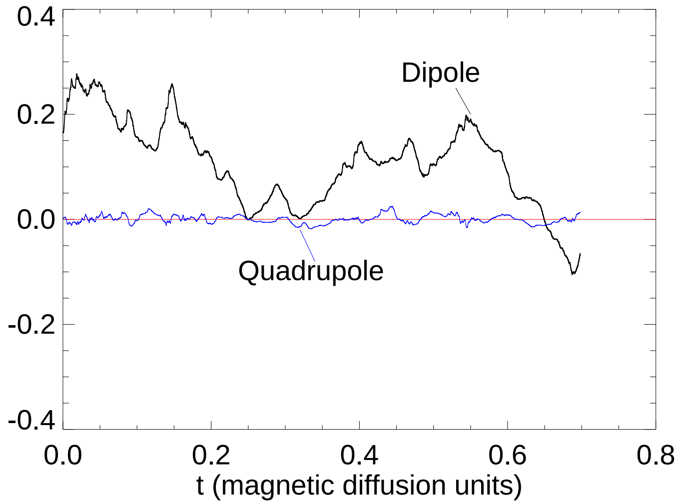
Left: radial surface magnetic field Right: radial velocity at $0.7R_J$

$E = 2.5 \times 10^{-5}$, $Ra = 2e07$, $Pr = 0.25$, $Pm = 3$. Pattern drifts, and convection fluctuates in time.

Rm is about 1000, for this run. Real Rm probably a hundred times larger. This solution has more small scale structure than currently known field, but real thing probably has even more small scale field.

Nests of convection a robust feature of this solution.

Maintaining the dipole



Dynamo maintains dipolarity for long times: diffusion time is 5×10^8 years. Excursions and a reversal, but went quadrupolar eventually.

Dipolar fields not easy to find. $Pr > 1$ gives dynamo wave type fields, with thermal wind important ($\alpha\omega$ behaviour).

Higher Ra gives small scale field (multipolar) with no significant dipole component. Lower Ra and the dynamo fails. Expect lower E to help, as then Ro can be small while $Rm > 1000$.

Compressible dynamos: good points, bad points

• In Boussinesq spherical shell dynamos, stable steady dipolar solutions are easy to find. Much harder in compressible models, both for polytropes and Jupiter-like models. Higher Rm needed for dynamo action, must have Ro small for dipolar solutions, so lower E required. Compressible dynamos are therefore more computationally demanding.

• Fortunately, anelastic code runs almost as fast as Boussinesq code (10% - 20% slower). However, whereas you can get nice dipolar dynamo fairly inexpensively at $E = 10^{-4}$, in compressible need to get to $\sim 10^{-5}$ to get even the basic dipolar dynamo.

• Code is nice and stable, easy to add source terms, change conductivity/density profiles. Spectral in θ and ϕ means accuracy increases fast with truncation level. 6th order finite differences on Chebyshev spacing gives good accuracy too (cf Benchmark).

• Currently parallelised only of the radial mesh points, i.e. number of processors = number of mesh points. This is probably not a fundamental problem.

• The big problem is the very small timestep needed. At $E = 10^{-5}$ its typically 10^{-7} of a magnetic diffusion time, which means millions of timesteps are needed. Behaviour can change fundamentally after a substantial fractioon of diffusion time, even though there may have been hundreds of turn-over times. This is what makes exploring the parameter space very time consuming.